

Instructions:

1. This test is closed book, but calculators may be used. Be sure to show your work and explain your reasoning. ("I mashed the Schur button" is not sufficient explanation.)
2. Time limit: 70 minutes.
3. Do all problems. They count equally.
4. Please start each problem on a new sheet of paper.

1. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

- a. Calculate the characteristic polynomial of the matrix A and use the Cayley-Hamilton Theorem to calculate A^{-1} .
- b. Either find a matrix S such that $S^{-1}AS$ is diagonal, or explain why no such S exists.

2. Let $B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

- a. Explain why you know immediately that B is diagonalizable.
- b. Find a diagonal matrix D and a matrix S such that $S^{-1}BS = D$.
- c. Is there an orthogonal matrix Q such that $Q^T BQ = D$. Why or why not?

3. Find the Singular Value Decomposition and pseudo inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \text{ and use the pseudo inverse to find the minimum norm least squares}$$

solution of $Ax = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 4 \\ 6 \end{pmatrix}$. Please find the "reduced SVD." I don't care about vectors that

don't matter in the end. You may express the pseudo inverse as the product of three matrices if you wish.

4. a. Show that the matrix $A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 6 \end{pmatrix}$ is positive definite.

b. Prove that any positive definite symmetric matrix is invertible and that its inverse is also positive definite.

5. Let A be a skew-symmetric matrix (i.e., entries are real and $A^T = -A$.)

a. Show that A is normal, and deduce that there is a unitary matrix U with $U^H A U$ diagonal.

b. What can you say about the eigenvalues of A ?

c. What can you say about the eigenvalues of e^{tA} ?

d. Must e^{tA} be unitary?

ANSWERS

1. a. The characteristic polynomial is $p(\lambda) = \lambda^3 - 6\lambda^2 + 12\lambda - 8$, and

$$A^{-1} = \frac{1}{8}(A^2 - 6A + 12I)$$

- b. **A** is not diagonalizable since the eigenvalue $\lambda = 2$ has algebraic multiplicity 2, but the eigenspace is only one dimensional.

2. a. **B** has three distinct eigenvalues.

$$b. S = \begin{pmatrix} 1 & -1 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- c. No, since **B** is not symmetric.

3. The singular value decomposition is

$$A = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The pseudoinverse is

$$A^+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and the minimum norm least squares solution is $A^+b = \begin{pmatrix} 1 \\ 1 \\ 5/2 \\ 5/2 \end{pmatrix}$.

4. a. One method is to show the upper left minors have positive determinant. Another is to row reduce and notice the pivots are all strictly positive.
b. If \mathbf{A} is positive definite, then the eigenvalues of \mathbf{A} are all strictly positive. Thus their reciprocals are all strictly positive. But these are the eigenvalues of the inverse of \mathbf{A} . Since they are all strictly positive, the inverse of \mathbf{A} is positive definite.
5. a. Just show that $A^H A = A A^H$ and use the spectral theorem.
b. Use $U^H A U = \Lambda$ diagonal to show that $\Lambda^H = \Lambda$. Thus the eigenvalues of \mathbf{A} are purely imaginary.
c. They are complex numbers of modulus 1.
d. Yes, since e^{tA} is normal and all eigenvalues have modulus 1.