Math 4317  
Final Examination  
17 December 1999

Instructions:

1. Please begin each problem on a new sheet of paper.
2. Please do any six of the seven problems. Please indicate clearly which problem you do not want graded. Be sure to explain your work.
3. This is an open book exam. You may use the text by Bartle (except for the Hints section and your class notes.

1. A real number $\alpha$ is said to be algebraic if there exists a polynomial $p$ with integer coefficients for which $p(\alpha) = 0$. In this problem you will prove that the set of algebraic numbers is countable.

   Define the weight of a polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$, $a_n \neq 0$ of degree $n$ to be $n + \sum_{j=0}^{n} |a_j|$.

   a. Show that for each natural number $k$, there are only finitely many polynomials with integer coefficients and weight $k$.

   b. Use the result of part a. to show that the set of algebraic numbers is countable.

2. If $A$ is a subset of $\mathbb{R}$, the derived set of $A$ is defined to be

   $$A' = \{ x : x \text{ is a cluster point of } A \}.$$ 

   a. Compute $A'$ where $A = [0,1]$ is the closed unit interval.

   b. With $B = \left\{ \frac{1}{n} + \frac{1}{m} : m,n \in \mathbb{N} \right\}$, compute $B'$, $B''$, and $B'''$. (I.e., the derived set of $B$, the derived set of the derived set of $B$, etc.)

   c. Prove that for any subset $A$ of $\mathbb{R}$, the derived set $A'$ is closed.

3. Sketch the set $\{(x,y) : |x| < y \}$ and prove, directly from the definition of compactness, that it is not a compact subset of $\mathbb{R}^2$. 

4. a. Suppose \( f \) is a real-valued function that is uniformly continuous on a set \( D \subset \mathbb{R} \). Prove that \( f \) can be extended to a function \( F \), defined and continuous on \( \overline{D} \), the closure of \( D \).

(That is, there is a function \( F \), defined and continuous on \( \overline{D} \), with \( F(x) = f(x) \) whenever \( x \in D \).

Suggestion: First show that if \( f \) is uniformly continuous on \( D \), and \( (x_n) \) is a Cauchy sequence in \( D \), then \( (f(x_n)) \) is a Cauchy sequence, and use this property to define \( F \).

b. Give an example of a bounded open set \( D \), and a bounded continuous function \( f \) defined on \( D \), which cannot be extended to a function \( F \), defined and continuous on \( \overline{D} \).

5. Determine which sequences converge, which diverge, and find the limit of each convergent sequence.

a. \( w_n = \left(1 + \frac{4}{n}\right)^n \)  

b. \( x_n = \left(1 + \frac{4}{n^2}\right)^n \)

c. \( y_0 = 7, \quad y_{n+1} = 5 - \frac{6}{y_n} \)  
d. \( z_n = \frac{1}{n^3} \sum_{j=1}^{n} j^2 \)

6. The Lebesgue Covering Theorem (page 77) asserts that if \( \{G_\alpha\} \) is a collection of open sets which covers a compact set \( K \subset \mathbb{R}^p \), then there exists a strictly positive number \( \lambda \) such that if \( x, y \in K \), and \( \|x - y\| < \lambda \), then there exists \( \alpha \) for which \( G_\alpha \) contains both \( x \) and \( y \).

a. Use the Lebesgue Covering Theorem to prove that a real-valued function that is continuous on a compact domain \( K \subset \mathbb{R} \) is uniformly continuous on \( K \).

b. Give an example of a function that is continuous but not uniformly continuous on a bounded subset of \( \mathbb{R} \).

7. Determine the radius of convergence of each of these power series. You may find Stirling's Formula (page 239-240) helpful. Stirling's formula, which may be used to approximate the factorial, is \( \lim \left( \frac{(n/e)^n \sqrt{2\pi n}}{n!} \right) = 1 \).

a. \( \sum_{n=0}^{\infty} \frac{n^2}{2^{3n}} x^n \)  
b. \( \sum_{n=0}^{\infty} \frac{n^2}{2^{3n}} x^{3n} \)

c. \( \sum_{n=0}^{\infty} \frac{n!}{n!} x^n \)  
d. \( \sum_{k=0}^{\infty} c_k x^k, \quad c_k = \frac{a(a-1)\cdots(a-k+1)}{k!} \)

The series in part d is known as the Binomial series. Be sure to consider both the cases \( a \in \mathbb{N} \) and \( a \notin \mathbb{N} \).