Instructions:

1. Please begin each problem on a new sheet of paper.
2. Please do any four of the five problems. Indicate clearly which problem you do not want graded. Be sure to explain your work.
3. This is a closed book exam. You may use one 8.5 by 11 inch sheet of notes.

1. Determine which of these sequences is convergent, which is divergent, and calculate the limit of each convergent sequence.

   a. \( s_n = \left(1 + \frac{1}{n}\right)^n \)
   b. \( x_n = \left(1 + \frac{1}{n^2}\right)^n \)
   c. \( y_1 = 6, \quad y_{n+1} = 4 - \frac{3}{y_n} \)
   d. \( z_n = \frac{1}{3n+1} + \frac{1}{3n+2} + \cdots + \frac{1}{5n} \)

2. Let \((x_n)\) be a sequence of real numbers, and define the sequence \((s_n)\) by \( s_n = \frac{1}{n} \sum_{k=1}^{n} x_k \).
   a. Show that if \((x_n)\) converges to \(x\), then the sequence of averages \((s_n)\) converges to \(x\) also.
   b. Give an example of a divergent sequence \((x_n)\) for which the sequence of averages \((s_n)\) converges.

3. Give an \(\varepsilon-\delta\) proof that \(f(x) = \sqrt{x}\) is continuous at \(a\) for each \(a > 0\).

4. Suppose \(f\) and \(g\) are functions from \(\mathbb{R}\) to \(\mathbb{R}\), and that both are continuous at \(a\). Provide an \(\varepsilon-\delta\) proof that the product \(fg\) is also continuous at \(a\).

5. Suppose \(f\) is a one-to-one function from \(\mathbb{R}^2\) onto \(\mathbb{R}\) (i.e., \(f\) is a bijection). Show that with \(D = \mathbb{R}^2 \setminus \{(0,0)\}, f(D)\) is not a connected subset of \(\mathbb{R}\). Deduce that there is no continuous one-to-one function from \(\mathbb{R}^2\) onto \(\mathbb{R}\).