Math 4320  
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Final Examination  
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Instructions:  
1. You may use the text by Churchill and your class notes on this exam.  
2. Please begin each problem on a new sheet of paper.  
3. Be sure to explain your work and justify your results.  
4. Problems count equally.

1. Determine all point(s) in the complex plane at which \( g(z) = z^2 \Re(z) \) is differentiable and calculate \( \frac{dg}{dz} \) at those point(s).

   Answer: \( G \) is differentiable only at 0. The derivative there is 0.

2. Calculate all values of 
   
a. \( i^i \).
   
b. \( \sin(\Log(i^i)) \). \( \Log \) denotes the principal branch of the logarithm.

   Answers:  
   a. \( e^{\left(\frac{\pi}{2} + 2\pi k\right)} \)
   
   b. -1.

3. Let \( f(z) = \frac{z^2}{z^6 + 64} \).

   a. In what domain is \( f \) analytic?
   
b. In what domain does the Taylor series for \( f \) centered at 0 converge?
   
c. In what domain does the Taylor series for \( f \) centered at 10i converge?
   
d. In what domain does the Taylor series for \( f \) centered at 1 converge?
   
e. Calculate any one of these series.

   Answers:  
   a. \( f \) is analytic except at the 6-th roots of -64.
   
   b. Disk of radius 2, center 0.
   
   c. Disk of radius 8, center 10i
   
   d. Disk of radius \( \sqrt{5 - 2\sqrt{3}} \), center 1
   
   e. \( \frac{1}{64} \sum_{n=0}^{\infty} (-1)^n \frac{z^{6n+2}}{2^{6n}} \)
4. Let \( f(z) = z \sin(\sqrt{z}) \). Calculate the Laurent series for \( f \), centered at 0, and the residue of \( f \) at 0. In what region does this series converge, and what type of singularity does \( f \) have at 0?

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}.
\]
Thus there is an essential singularity at 0, with residue 0. The series converges for any \( z \neq 0 \).

5. Let \( f(z) = \frac{\sin(z)}{z^6(z^2-1)} \).

a. Calculate the residue of \( f \) at 0 by calculating at least a few terms of the appropriate Laurent series for \( f \) at 0.

b. How many Laurent series are there for \( f \) centered at 0? In what regions do they converge to \( f \)?

c. Find the residue of \( f \) at 1.

\[
\sin(1) \frac{1}{2}.
\]

6. Use the residue theorem to show that \( \int_{0}^{\infty} \frac{dz}{z^6 + 1} = \frac{\pi}{3} \).

7. Use the residue theorem to calculate \( \int_{0}^{2\pi} \frac{d\theta}{2 + \cos(\theta)} \).

Answer: \( \frac{2\pi}{\sqrt{3}} \).