Instructions:  
1. You may use the text by Churchill and notes on one 8.5” by 11” sheet of paper.
2. Please begin each problem on a new sheet of paper.
3. Be sure to explain your work and justify your results.
4. Problems count equally.

1. How many roots does $6z^4 + z^3 - 2z^2 + z - 1$ have in the disk $|z| < 1$?

2. Use the theory of residues to find the inverse Laplace transform of 
   \[ F(s) = \frac{1}{(s + 1)^2(s^2 + 2s + 2)}. \]

3. Using techniques analogous to those we used to prove the Poisson Integral Formula for the disk, one can deduce the Poisson Integral Formula for the half plane. Specifically, if $F$ is a bounded, piecewise smooth function on the real line, then 
   \[ U(x, y) = \frac{1}{\pi} \int_{\epsilon} \frac{yF(t)}{(t - x)^2 + y^2} dt \]
   is harmonic in the upper half plane and 
   \[ \lim_{y \to 0^+} U(x, y) = F(x) \]
   at points $x$ where $F$ is continuous.

   Use this formula to find a function $U(x, y)$, harmonic in the upper half plane, with 
   \[ U(x, 0) = \begin{cases} 
   0 & |x| > 1 \\
   1 & |x| < 1 
   \end{cases} \]

   (Please note that this problem was solved by different means in the text and in class, but that two correct solutions may appear different at first glance. Please express your answer in some reasonably simple form but do not try to make it look exactly as obtained before.)

4. Verify that $u(x, y) = x^3 - 3xy^2 + 2xy$ satisfies Laplace’s equation, and then find a harmonic function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is analytic.
Answers.

1. 4, by Rouche’s theorem.

2. \( f(t) = te^{-t} - e^{-t}\sin(t) \)

3. The integral may done with a trig substitution, yielding
\[
U(x,y) = \frac{1}{\pi} \left( \arctan \left( \frac{1 - x}{y} \right) - \arctan \left( \frac{-1 - x}{y} \right) \right)
\]

4. \( v(x,y) = 3x^2y - x^2 - y^3 + y^2 \)