Heat Equation Examples

Generation and Convection

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Math 5581

A. D. Andrew

Example 1 - HE with generation

Heat equation with generation.

\[ u_{xx} + 1 = u_t \quad 0 < x < 1, \ 0 < t \]
\[ u(0,t) = 0 \]
\[ u(1,t) = 0 \]
\[ u(x,0) = 0 \]

The time independent solution is

\[ v := x \rightarrow -\frac{1}{2} x^2 + \frac{1}{2} x; \]

\[ v := x \rightarrow -\frac{1}{2} x^2 + \frac{1}{2} x \quad (1.1) \]

so we now let \( w = u - v(x) \), and \( w \) satisfies

\[ w_{xx} = w_t \]
\[ w(0,t) = w(1,t) = 0 \]
\[ w(x,0) = -v(x) \]

The Fourier coefficients are

\[ a := n \rightarrow 2 \cdot \operatorname{int} \left( \frac{1}{2}, x^2 - x \cdot \sin(n \cdot \pi \cdot x), x=0..1 \right); \]

\[ a := n \rightarrow 2 \int_0^1 \frac{1}{2} \left( x^2 - x \right) \sin(n \pi x) \, dx \quad (1.2) \]

\[ a(n); \]
\[ \frac{-2 + n \pi \sin(n \pi) + 2 \cos(n \pi)}{n^2 \pi^3} \]  

(1.3)

and a partial sum for \( w \) is

\[ wIN := (x, t, N) \rightarrow \text{add}(a(n) \cdot \exp(-n^2 \pi^2 t) \cdot \sin(n \pi x), n = 1 \ldots N) \; \]

(1.4)

and a partial sum for the solution to the original problem is

\[ uIN := (x, t, N) \rightarrow \nu(x) + wIN(x, t, N) \; \]

(1.5)

Here are pictures

\[ \text{plot}([\text{seq}(uIN(x, k \cdot .01, 25), k=0..4)], x=0..1, \text{color} = \text{black}); \]

\[ \text{plot}([\text{seq}(uIN(x, k \cdot 1, 25), k=0..4)], x=0..1, \text{color} = \text{black}); \]
> plot([v(x), u1N(x, 3, 25)], x = 0 .. 1, color = black);
\textbf{Example 2 - HE with convective BC}

\[ u_{xx} = u_t \quad 0 < x < 1, \quad 0 < t \]
\[ u(0,t) = 50 \]
\[ -u_x(1,t) = (u(1,t) - 100) \]
\[ u(x,0) = 0 \]

The steady state solution is

\[ v_2 := x \rightarrow 25 \cdot x + 50; \]

\[ v_2 := x \rightarrow 25 \cdot x + 50 \quad (2.1) \]

Subtracting the steady-state solution, we obtain the problem with homogeneous boundary values

\[ w_{xx} = w_t \]
\[ w(0,t) = 0 \]
\[ w(1,t) + wx(1,t) = 0 \]
\[ w(x,0) = g(x) \]

with
\[ g^2 := x \rightarrow -v^2(x); \]
\[ g^2 := x \rightarrow -v^2(x) \tag{2.2} \]

The eigenvalues are \(-\lambda n^2\) and eigenfunctions are as follows. The first few Lammbdas are found in our text, the rest are from Abramowitz and Stegun.

\[ X := (x, n) \rightarrow \sin(\lambda(n) \cdot x); \]
\[ X := (x, n) \rightarrow \sin(\lambda(n) x) \tag{2.3} \]

\[ \lambda(1) := 2.0288; \lambda(2) := 4.9132; \]
\[ \lambda(3) := 7.9787; \lambda(4) := 11.0855; \]
\[ \lambda(5) := 14.2074; \lambda(6) := 17.3364; \]
\[ \lambda(7) := 20.4692; \lambda(8) := 23.6043; \]
\[ \lambda(9) := 26.7409; \tag{2.4} \]

The coefficients of the series solution for \(w\) are

\[ b_2 := n \rightarrow \frac{\text{evalf} \left( \int g^2(x) \sin(\lambda(n) x) \, dx \right)}{\int_0^1 \sin(\lambda(n) x)^2 \, dx} \]
\[ b_2 := n \rightarrow \frac{\text{evalf} \left( \int_0^1 g^2(x) \sin(\lambda(n) x) \, dx \right)}{\int_0^1 \sin(\lambda(n) x)^2 \, dx} \tag{2.5} \]

and the 9th partial sum is

\[ w_2 := (x, t) \rightarrow \text{add} \left( b_2(n) \cdot X(x, n) \cdot \exp \left( -\lambda(n)^2 \cdot t \right), n = 1..9 \right); \]
\[ w_2 := (x, t) \rightarrow \text{add} \left( b_2(n) \cdot X(x, n) \cdot \exp \left( -\lambda(n)^2 t \right), n = 1..9 \right) \tag{2.6} \]
The 9th partial sum for u is

\[ u^2 := (x, t) \rightarrow v2(x) + w2(x, t); \]

\[ u^2 := (x, t) \rightarrow v2(x) + w2(x, t) \] \hspace{1cm} (2.7)

As usual, here are pictures showing the solution for various small values of t, followed by the steady state solution.

\[ \text{plot( [seq(u2(x, k*0.075), k = 0..10)], x = 0..1, color = black);} \]