Suppose that \( f \) is analytic in the annulus \( r < |z - z_0| < R \). Then \( f \) can be expanded in a Laurent series

\[
f(z) = \cdots + \frac{a_{-k}}{(z - z_0)^k} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots
\]

where

\[
a_k = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{k+1}} \, dz \quad \text{for } k = 0, \pm 1, \pm 2,
\]

and where \( \Gamma \) is any positively oriented simple closed contour in the annulus which encloses \( z_0 \). Let us take for granted that the convergence properties of the Laurent series allow a term by term integration. If \( \Gamma \) is a simple closed curve around \( z_0 \) in the annulus then

\[
\int_{\Gamma} f(z) \, dz = \sum_{k=-\infty}^{\infty} \int_{\Gamma} a_k(z - z_0)^k \, dz = 2\pi i a_{-1}
\]

since all other terms of the series integrate to zero. However, the coefficient \( a_{-1} \) is generally not known, and if one had to find it from its integral representation then one may as well integrate \( f \) directly.

The usefulness of the Laurent expansion increases if the point \( z_0 \) is the center of a disk and \( f \) is analytic at every other point of the disk. Hence from now on we shall assume that

\( f \) is analytic for \( 0 < |z - z_0| < R \).

If \( f \) is not analytic at \( z_0 \) then we say that \( z_0 \) is an isolated singularity of \( f \). If we now consider the Laurent series of \( f \) in the punctured disk \( 0 < |z - z_0| < R \) we have

**Definition:** The coefficient \( a_{-1} \) is the residue of \( f \) at \( z_0 \).

We use the notation

\[
a_{-1} = \text{Res}(f, z_0).
\]

If \( f \) is analytic at \( z_0 \) then \( \text{Res}(f; z_0) = 0 \). In general, only those points will be of interest where \( f \) is singular. It is generally not difficult to find the points where \( f \) is singular,
although the residue may not always be easy to find. We had seen that if $\Gamma$ is a simple closed curve around $z_0$ in $|z - z_0| < R$ then

$$\int_{\Gamma} f(z)dz = 2\pi i \text{Res}(f : z_0).$$

Suppose now that we need to find

$$\int_{\Gamma} f(z)dz$$

around some simple closed curve $\Gamma$ enclosing a domain $D$ where $f$ has $N$ isolated singularities at the points $z_j$, $j = 1, 2, \ldots, N$. Away from these points and on $\Gamma$, $f$ is assumed to be analytic. Around each $z_j$ there is an annulus where $f$ is analytic so that in this annulus $f$ can be written in terms of a Laurent series around $z_j$. Moreover, the integral around $\Gamma$ can be deformed into $N$ circles around the $\{z_j\}$ and each circle integrates into $2\pi i \text{Res}(f; z_j)$. Thus we have the so-called residue theorem:

Let $\Gamma$ be a simple closed positively oriented curve.

Assume that $f$ is analytic on $\Gamma$ and inside $\Gamma$ except at isolated points $\{z_j\}$, $j = 1, \ldots, N$. Then

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_{j=1}^{N} \text{Res}(f ; z_j).$$

It is not obvious at this time that this theorem is useful because we appear to need to know the Laurent series before we know the residue. However, it usually is possible to classify the singularities of $f$ and to find the residues with simple calculations. Suppose that

$$f(z) = \sum_{k=\infty}^{\infty} a_k(z - z_0)^k \quad \text{for } 0 < |z - z_0| < R$$

then we have

**Definition:** If $a_{-m} = 0$ for all $m > M$, $a_{-M} \neq 0$ for $M \geq 1$ then $f$ has a pole of order $M$ at $z_0$. If $a_m = 0$ for all $m < M$, $a_M \neq 0$ for $M \geq 1$ then $f$ has a zero of order $M$ at $z_0$.

It can be shown that if $f$ has a pole of order $M$ then there exists an analytic function $g$ such that

$$f(z) = \frac{g(z)}{(z - z_0)^M}$$
where \( g(z_0) \neq 0 \). Similarly, if \( f \) has a zero of order \( M \) then
\[
f(z) = (z - z_0)^M g(z)
\]
where \( g(z) \) is analytic and does not vanish at \( z_0 \). For example, the function
\[
f(z) = \frac{1}{(z^2 - 1)} = \frac{1}{(z - 1)(z + 1)}
\]
has a pole of order 1 (also called a simple pole) at \( z = \pm 1 \). Similarly, the function
\[
f(z) = \frac{\sin z}{\cos z - 1}
\]
has a simple pole at \( z = 0 \) which may be inferred from the Taylor series
\[
\sin z = z - z^3/3! + z^5/5! \cdots
\]
\[
\cos z = 1 - z^2/2! + z^4/4! \cdots
\]
and writing
\[
f(z) = \frac{z(1 - z^2/3! + z^4/5! - \cdots)}{z^2(-1/2! + z^2/4! - \cdots)} = \frac{1}{z} g(z)
\]
Let us now suppose that \( f \) has a pole of order \( m \) at \( z_0 \). Then
\[
(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{m-1} + a_0(z - z_0)^m + \cdots
\]
from which follows that
\[
\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] = (m - 1)!a_{-1}.
\]
For example, the function
\[
f(z) = \frac{z^3 - 2iz + 5}{(z^2 - 2z + 1)}
\]
has a pole of order 2 at \( z = 1 \). Hence the residue is
\[
a_{-1} = \lim_{z \to 1} \frac{d}{dz} \left[ \frac{(z - 1)^2(z^3 - 2iz + 5)}{z^2 - 2z + 1} \right] = 3 - 2i.
\]
In many instances only simple poles appear in which case
\[
a_{-1} = \lim_{z \to z_0} (z - z_0) f(z).
\]
Not all singularities are poles of order \( m \). For example, the function

\[
f(z) = \frac{z}{z}
\]

has a so-called removable singularity at \( z = 0 \). Its Laurent series in \( 0 < |z - 0| < \infty \) is the one term expansion

\[
f(z) = 1 \text{ on } 0 < |z|.
\]

On the other end of the spectrum are those functions whose Laurent expansion does not terminate at some finite \(-m\). For example, since

\[
e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!} \quad \text{for all } w,
\]

it follows that

\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} \quad \text{on } |z| > 0.
\]

When there are non-zero coefficients in the Laurent expansion for infinitely many negative integers then the singularity is called an essential singularity.
Module 23/Module 24 - Homeworks

1) Find the Taylor series for the following functions around the given point \( z_0 \) and give the radius of convergence.
   i) \( f(z) = e^z, \quad z_0 = 0 \)
   ii) \( f(z) = \sin z, \quad z_0 = 0 \)
   iii) \( f(z) = \cos z, \quad z_0 = 0 \)
   iv) \( f(z) = \log z, \quad z_0 = 1 \)
   v) \( f(z) = 1/(1 + z^2), \quad z_0 = 1 \).

2) Find the Laurent expansion of
   \[
   f(z) = \frac{1}{z(z + 1)}
   \]
   i) which is valid for \( 0 < |z| < 1 \),
   ii) which is valid for \( |z| > 1 \).
   iii) Find \( \text{Res}(f; 0) \).

3) Evaluate
   \[
   \int_{|z|=2} \frac{\sin^2 z}{(i + z)^2} \, dz.
   \]

4) Let \( D \) be the triangle in the complex plane with vertices at \(-1 + 5i, -3 + i, 1 - i\). Let \( \Gamma \) be the positively oriented boundary of \( D \). Compute
   i) \( \int_{\Gamma} \frac{e^z}{(z^2 + 49)^2(z - 1)} \, dz \)
   ii) \( \int_{\Gamma} \frac{e^z}{(z^2 + 49)^2 z} \, dz \)

5) Let \( f \) and \( g \) be analytic on \( |z - z_0| < r \) and suppose that \( f(z_0) = g(z_0) = 0 \). Prove l'Hospital’s rule:
   \[
   \lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}
   \]
   provided the limit exists.