MODULE 25

Topics: Application of the residue theorem

Complex contour integrals occur in the computation of inversion formulas for Fourier and Laplace transforms, but they may also be used to evaluate certain definite real integrals.

1) Integration of trigonometric functions:

Suppose we are to compute

$$\int_{0}^{2\pi} F(\sin \theta, \cos \theta) d\theta.$$ 

Since

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad z = e^{i\theta} \quad \text{for any} \ z \text{ on } |z| = 1$$

we find that

$$\sin z = \frac{z - 1/z}{2i},$$

and analogously

$$\cos z = \frac{z + 1/z}{2}.$$

Moreover, from $z = e^{i\theta}$ follows that

$$dz = ie^{i\theta} d\theta = iz d\theta.$$

Hence the original integral is a parametrization of the integral

$$\int_{|z|=1} F \left( \frac{z - 1/z}{2i}, \frac{z + 1/z}{2} \right) \frac{dz}{iz}.$$ 

But the complex integral is equal to $2\pi i$ times the residues of the integrand inside the unit circle.

**Illustration:** Suppose $b$ is real and $|b| < 1$. Then

$$\int_{0}^{2\pi} \frac{d\theta}{1 + b \sin \theta} = \int_{|z|=1} \frac{1}{1 + \frac{b(z - 1/z)}{2i}} \frac{dz}{iz} = 2 \int_{|z|=1} \frac{dz}{2iz + bz^2 - b}.$$ 

The roots of the denominator for $|b| < 1$ are

$$z_1 = -\frac{i \left( 1 + \sqrt{1 - b^2} \right)}{b}.$$
and
\[ z_2 = -\frac{i(1 - \sqrt{1-b^2})}{b}. \]

By inspection \(|z_1| > 1\) so the simple pole at \(z_1\) is outside the unit circle and does not influence the integral. If we set \(\alpha^2 = 1 - b^2\) then we can write
\[ z_2 = i\frac{\sqrt{1-\alpha}}{\sqrt{1+\alpha}} \]

which tells us that the integrand has a simple pole inside the unit circle at \(z_2\). We compute with l’Hospital’s rule
\[ \text{Res}(f, z_2) = \lim_{z \to z_2} \frac{z - z_2}{2iz + b z^2 - b} = \frac{1}{2i + 2bz_2} = \frac{1}{2i\sqrt{1-b^2}} \]

where \(f\) denotes the integrand \(1/(2iz + bz^2 - b)\). Thus
\[ \int_0^2 \frac{1}{1 + b \sin \theta} \, d\theta = 2\pi i \text{ Res}(f : z_2) = \frac{2\pi}{\sqrt{1-b^2}}. \]

The integral is real so the computation would have to be wrong if the final result were not real.

2) Integration of rational functions along the line.

Let us consider the improper integral
\[ \int_{-\infty}^{\infty} \frac{P_M(x)}{Q_N(x)} \, dx \]

where \(P\) and \(Q\) are polynomials of degree \(M\) and \(N\). We shall assume that \(Q_N(x) \neq 0\) for any \(x \in (-\infty, \infty)\) and
\[ N \geq M + 2. \]

These conditions assure that the integral exist and that it may be obtained from its principal value
\[ \int_{-\infty}^{\infty} \frac{P_M(x)}{Q_N(x)} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{P_M(x)}{Q_N(x)} \, dx. \]

Let us now consider the domain \(D\) in the upper half plane bounded by the semi-circle \(z = Re^{i\theta}, \theta \in [0, \pi]\) and the \(x\)-axis. Let \(\Gamma\) be the positively oriented boundary of \(D\). Then
\[ \int_{\Gamma} \frac{P_M(z)}{Q_N(z)} \, dz = 2\pi i \sum \text{Res}(f : z_j) \]

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where the residues have to be found at the poles of \( f(z) = P_M(z)/Q_N(z) \). As in the proof of the fundamental theorem of algebra we know from

\[
Q_N(z) = z^N (a_N + a_{N-1}/z + \cdots + a_0/z^N)
\]

that the roots of \( Q_N(z) = 0 \) have to lie inside some circle with radius \( R_0 \). Hence if in our contour integral \( R \) is sufficiently large then \( \Gamma \) will enclose all singularities of \( f \) in the upper half plane. If we write

\[
\int_{\Gamma} f(z)dz = \int_{-R}^{R} f(x)dx + \int_{\gamma} f(z)dz = 2\pi i \sum \text{Res}(f : z_j)
\]

where \( \gamma \) is the semi-circle then it follows that the left hand side is independent of \( R \) for all \( R > R_0 \) since all residues are now inside \( D \). Finally, by hypothesis we have on

\[
\left| \frac{P_M(z)}{Q_N(z)} \right| > \frac{K}{|z|^2} \quad \text{for sufficiently large } R \text{ and some constant } K.
\]

It follows that

\[
\left| \int_{\gamma} f(z)dz \right| < \left( \frac{K}{R^2} \right) 2\pi R \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\]

Thus

\[
\int_{-\infty}^{\infty} \frac{P_M(x)}{Q_N(x)} dx = 2\pi i \sum \text{Res} \left( \frac{P_M(z)}{Q_N(z)} : z_j \right)
\]

where the \( z_j \) are the roots of \( Q_N(z) = 0 \) in the upper half plane.

**Illustration:** Consider

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^6} dx.
\]

In this case \( M = 0 \) and \( N = 6 \) and \( Q \) does not vanish on the line. Hence the formula for rational functions applies. We need the zeros of \( Q \). We know that the equation

\[
1 + z^6 = 0
\]

has the six solutions

\[
z_k = e^{i(\pi + 2\pi k)/6}, \quad k = 0, 1, 2, 3, 4, 5.
\]

The first three roots lie in the upper half plane. The other three are the complex conjugates.

There are three simple poles in the upper half plane and the residues are

\[
\lim_{z \to z_k} \left[ \frac{(z - z_k)}{1 + z^6} \right] = \frac{1}{6z_k^5}.
\]
It follows that

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^6} \, dx = \frac{2\pi i}{6} \left[ e^{-5\pi i/6} + e^{-\pi i/6} + e^{-2\pi i/6} \right]
\]

\[
= \frac{2\pi i}{6} \left[ e^{-5\pi i/6} + e^{-\pi i/2} - e^{5\pi i/6} \right]
\]

\[
= \frac{2\pi i}{6} \left[ -2i \sin(5\pi/6) - i \right] = \frac{2\pi}{3}.
\]