MODULE 4

Topics: Orthogonal projections

**Definition:** Let $V$ be an inner product space over $F$. Let $M$ be a subspace of $V$. Given an element $y \in V$ then the orthogonal projection of $y$ onto $M$ is the vector $Py \in M$ which satisfies

$$y = Py + v$$

where $v$ is orthogonal to every element $m \in M$ (in short: $v$ is orthogonal to $M$).

The orthogonal projection, if it exists, is uniquely defined because if

$$y = Py_1 + v_1$$

and

$$y = Py_2 + v_2$$

then by subtracting we find that

$$Py_1 - Py_2 = -(v_1 - v_2).$$

Since $Py_1 - Py_2$ is an element of $M$ and $(v_1 - v_2)$ is orthogonal to $M$ it follows from

$$\langle Py_1 - Py_2, Py_1 - Py_2 \rangle = -\langle v_1 - v_2, Py_1 - Py_2 \rangle = 0$$

that $Py_1 = Py_2$.

The existence of orthogonal projections onto infinite dimensional subspaces $M$ is complicated. Hence we shall consider only the case where

$$M = \text{span}\{x_1, \ldots, x_n\}$$

where, in addition, the vectors $\{x_1, \ldots, x_n\}$ are assumed to be linearly independent so that the dimension of $M$ is $n$. In this case $Py$ must have the form

$$Py = \sum_{j=1}^{n} \alpha_j x_j.$$
for some properly chosen scalars \( \{\alpha_j\} \). From the definition of the orthogonal projection now follows that

\[
\langle y, m \rangle = \left( \sum_{j=1}^{n} \alpha_j x_j, m \right) + \langle v, m \rangle = \sum_{j=1}^{n} \alpha_j \langle x_j, m \rangle
\]

for arbitrary \( m \in M \). In particular, it has to be true for \( m = x_i \) for each \( i \), and if it is true for each \( x_i \) then it is true for any linear combination of the \( \{x_j\} \), i.e., it is true for all \( m \in M \). Hence the orthogonal projection is

\[
Py = \sum_{j=1}^{n} \alpha_j x_j
\]

where the \( n \) coefficients \( \{\alpha_j\} \) are determined from the \( n \) equations

\[
\langle y, x_i \rangle = \sum_{j=1}^{n} \alpha_j \langle x_j, x_i \rangle, \quad i = 1, \ldots, n.
\]

In other words, the coefficients \( \{\alpha_j\} \) are found from the matrix system

\[
A\tilde{\alpha} = b, \quad \tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)
\]

where

\[
A_{ij} = \langle x_j, x_i \rangle
\]

and

\[
b_i = \langle y, x_i \rangle.
\]

The question now arises: Does the orthogonal projection always exist? Or equivalently, can I always solve the linear system

\[
A\tilde{\alpha} = b.
\]

The solution \( \tilde{\alpha} \) exists and is unique whenever \( A \) is invertible, or what is the same, whenever

\[
A\tilde{\beta} = 0
\]

has only the zero solution \( \tilde{\beta} = (0, \ldots, 0) \). Suppose that there is a non-zero solution \( \tilde{\beta} = (\beta_1, \ldots, \beta_n) \). If we set

\[
w = \sum_{j=1}^{n} \beta_j x_j
\]
then we see by expanding the inner product that
\[ \langle w, w \rangle = A\bar{\beta} \cdot \bar{\beta}. \]
But this implies that \( w = 0 \) which contradicts that the \( \{x_j\} \) are linearly independent. Hence \( A\bar{\beta} = 0 \) cannot have a non-zero solution. \( A \) is invertible and the orthogonal projection is computable.

**Examples:**
1) Let \( M = \text{span}\{(1, 2, 3), (3, 2, 1)\} \in \mathbb{R}^3 \). We see that \( M \) is the plane in \( \mathbb{R}^3 \) (through the origin, of course) given algebraically by
\[
x_1 - 2x_2 + x_3 = 0.
\]
Then the projection of the unit vector \( \hat{e}_1 = (1, 0, 0) \) onto \( M \) is the vector
\[
P\hat{e}_1 = \alpha_1 (1, 2, 3) + \alpha_2 (3, 2, 1)
\]
where \( \alpha_1 \) and \( \alpha_2 \) are found from
\[
\begin{pmatrix}
\langle x_1, x_1 \rangle & \langle x_2, x_1 \rangle \\
\langle x_1, x_2 \rangle & \langle x_2, x_2 \rangle
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2
\end{pmatrix}
= \begin{pmatrix} 14 & 10 \\ 10 & 14 \end{pmatrix}
\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
= \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\]
It follows that
\[
P\hat{e}_1 = -\frac{1}{6} (1, 2, 3) + \frac{1}{3} (3, 2, 1) = (\frac{5}{6}, \frac{1}{3}, -\frac{1}{6}).
\]
We note that in this case \( v = \hat{e}_1 - P\hat{e}_1 = (\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}) \) which is perpendicular to the plane \( M \) as required.

2) Find the orthogonal projection of the function \( f(t) = t^3 \) onto
\[
M = \text{span}\{1, t, t^2\}
\]
when \( V = C^0[-1, 1] \) and \( \langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt \). In other words, find the projection of \( t^3 \) onto the subspace of polynomials of degree \( \leq 2 \).

Answer:
\[
P(t^3) = \alpha_0 + \alpha_1 t + \alpha_2 t^2
\]
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where
\[
\begin{pmatrix}
\langle 1, 1 \rangle & \langle t, 1 \rangle & \langle t^2, 1 \rangle \\
\langle 1, t \rangle & \langle t, t \rangle & \langle t^2, t \rangle \\
\langle 1, t^2 \rangle & \langle t, t^2 \rangle & \langle t^2, t^2 \rangle
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
\langle t^3, 1 \rangle \\
\langle t^3, t \rangle \\
\langle t^3, t^2 \rangle
\end{pmatrix}.
\]

Carrying out the integrations we find the algebraic system
\[
\begin{pmatrix}
2 & 0 & 2/3 \\
0 & 2/3 & 0 \\
2/3 & 0 & 2/5
\end{pmatrix}
\begin{pmatrix}
\alpha_0 \\
\alpha_1 \\
\alpha_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
2/5 \\
0
\end{pmatrix}
\]

from which we obtain the orthogonal projection
\[Pt^3 = \frac{3}{5} t\]

The question now arises. Why should one be interested in orthogonal projections?
Module 4 - Homework

1) Find the orthogonal projection with respect to the dot product of the vector \( y = (0, 2) \) onto 

\[ M = \text{span}\{(1, 1)\} \]

and draw a picture that makes clear what is

i) \( M \)

ii) \( y \)

iii) \( Py \)

iv) \( v \).

2) Let \( A = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \) and define for \( \mathbb{R}^2 \)

\[ \langle x, y \rangle = Ax \cdot y. \]

i) Show that \( \langle , \rangle \) is an inner product on \( \mathbb{R}^2 \).

ii) Find all vectors which are orthogonal to the vector \( (0, 1) \) with respect to this inner product.

iii) Compute the orthogonal projection of the vector \( (0, 2) \) onto \( M = \text{span}\{(1, 1)\} \).

Draw a picture as in Problem 1.

3) Let \( V = C^0[0, 2\pi], \langle f, g \rangle = \int_0^{2\pi} f(t)g(t)dt \). Find the orthogonal projection of \( f(t) \equiv t \) onto \( M = \text{span}\{1, \cos t, \cos 2t\} \).

4) Find the orthogonal projection \( Pf \) in \( L_2(0, 1) \) of \( f(t) \equiv \cos(t - 4) \) onto 

\[ M = \text{span}\{\cos t, \sin t\}. \]

Compute

\[ v = f - Pf. \]

5) Let 

\[ M_1 = \text{span}\{(1, 2, 1, 2), (1, 1, 2, 2)\} \]

\[ M_2 = \text{span}\{(1, 2, 1, 2), (1, 1, 2, 2), (1, 0, 3, 2)\}. \]

Let \( y = (1, 0, 0, 0) \).

i) With respect to the dot product find the orthogonal projections of \( y \) onto \( M_1 \) and \( M_2 \).
ii) Show that $M_1 = M_2$.

6) Suppose a pole of height $5m$ stands vertically on a hillside. The elevation of the ground relative to the base of the pole is $+6m$ at a distance of $50m$ to the east and $-17m$ at a distance $200m$ south from the pole. Suppose the sun is in the southwest and makes an angle of $\pi/6$ with the pole. Assume that the hillside can be approximated by a plane. Find the vector which describes the shadow of the pole on the ground. What is its length?