Homework assignments for Math 4221 Spring 2009

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1 Due by February 3 2009

1. Consider the Bernoulli scheme for some $n \in \mathbb{N}$, $p \in (0,1)$ and $q = 1-p$:

$$\begin{aligned} \Omega &= \{(\omega_1, \ldots, \omega_n) : \quad \omega_k = 0 \ \text{ or } 1 \text{ for all } k\}, \\ p(\omega) &= (p^{\omega_1} q^{1-\omega_1}) \ldots (p^{\omega_n} q^{1-\omega_n}), \\ X_k(\omega) &= \omega_k, \end{aligned}$$

   Prove that $\mathsf{P}\{X_2 = 1\} = p$.

2. For the previous problem, prove that $X_1, \ldots, X_n$ are jointly independent.

3. Let $X_1, X_2, \ldots$ be Bernoulli random variables with parameter $p \in (0,1)$. Let $S_n = X_1 + \ldots + X_n$. Find $\mathsf{P}\{S_3 = 2\}$.

4. Let us consider a tetrahedral die. Three of its faces are painted Red, Green, and Blue, respectively, and the fourth is painted with all three colors. We roll the die once and look at the bottom face. We define

$$X_R = \begin{cases} 1, & \text{there is Red on the bottom face} \\ 0, & \text{there is NO Red on the bottom face} \end{cases},$$

   and random variables X_G, X_B are defined similarly. We assume that each of the four faces has probability $1/4$. Prove that two random variables in each pair (X_R, X_G), (X_R, X_B), and (X_G, X_B) are independent of each other. Prove that X_R, X_G, X_B are not jointly independent.

5. Let $X_1, X_2, \ldots$ be Bernoulli random variables with parameter $p \in (0,1)$. Let $S_n = X_1 + \ldots + X_n$. Prove that S_3 is not independent of X_1.

6. Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with

$$\mathsf{P}\{X_1 = 1\} = p, \quad \mathsf{P}\{X_1 = -1\} = q.$$

Let S_n be the random walk generated by X:

$$S_n = X_1 + \ldots + X_n.$$

For any n, find a formula for $\mathsf{P}\{S_n = 0\}$.

7. Let $X_1, X_2, \ldots$ be i.i.d. r.v.'s with

$$\mathsf{P}\{X_1 = 1\} = \mathsf{P}\{X_1 = 0\} = \frac{1}{2}.$$

Let S_n be the random walk generated by X:

$$S_n = X_1 + \ldots + X_n.$$

Use Local Central Limit Theorem to approximate

$$\frac{\mathsf{P}\{S_{1000} = 500\}}{\mathsf{P}\{S_{1000} = 520\}}.$$

8. Find the variance of a random variable uniformly distributed on $\{1, \ldots, 10\}$.

9. Use Chebyshev's inequality to estimate

$$\mathsf{P}\{|X_1 - X_2| > 5\}$$

if $\mathsf{E}X_1 = \mathsf{E}X_2 = 1$, $\mathsf{E}X_1^2 = 2$, $\mathsf{E}X_2^2 = 20$.

10. In the setting of Problem 6, let us take any whole numbers $A < B$ and for each $x \in \{A, A+1, \ldots, B\}$ define $S_n^x = x + S_n$. Let

$$\beta(x) = \mathsf{P}\{\text{random walk } S^x \text{ reaches level } B \text{ before reaching level } A\}.$$

Draw a sketch of the graph of $\beta(x)$ for three cases (a)$p > q$, (b)$p = q = 1/2$, (c)$p < q$.

11. In the setting the previous problem, fix B and study the limiting behaviour of $\beta(x)$ as $A \to -\infty$.

2 Due by February 24 2008

In problems 1 to 4 we consider a symmetric random walk: $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. r.v.'s with $\mathsf{P}\{X_1 = 1\} = \mathsf{P}\{X_1 = -1\} = 1/2$, and $S_n = X_1 + \ldots + X_n$ for all n. We also set $S_0 = 0$.

1. Use Local Central Limit Theorem to derive

$$\mathsf{P}\{S_{2k} = 0\} \sim \frac{1}{\sqrt{\pi k}}, \quad k \to \infty.$$

2. Use the reflection principle to show that for any $N > 0$

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k \ge N, \quad S_n < N\right\} = \mathsf{P}\{S_n > N\}.$$

3. Use the reflection principle to show that for any $N > 0$,

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k \ge N\right\} = 2\mathsf{P}\{S_n \ge N\} - \mathsf{P}\{S_n = N\}.$$

4. Use the previous problem to show that for any $N > 0$,

$$\mathsf{P}\left\{\max_{1\le k\le n} S_k = N\right\} = \mathsf{P}\{S_n = N\} + \mathsf{P}\{S_n = N + 1\}.$$

5. Exercise 1.1

6. Exercise 1.2

7. Exercise 1.3

8. Prove that for a Markov chain with finitely many states there is at least one state that is not inessential. Is this true for Markov chains with infinitely many states?

9. For each of the following Markov transition matrices, (i) draw the associated transition graph; (ii) find classes of communicating states and inessential states; (iii) for each class find its period; (iv) find all stationary distributions; (v) for each initial state consider the Markov chain with given transition probabilities and originatig at that state; does the distribution of this Markov chain at time n have a limit as $n \to \infty$? do the limits coincide for different initial states? If your answer to at least one of these questions is "no", explain why this does not contradict the Perron–Frobenius theorem.

(a) "Symmetric random walk with absorption"

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) "Symmetric random walk with reflection"

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

10. Exercise 1.5

11. Exercise 1.6

12. Exercise 1.8

13. Exercise 1.9

14. Exercise 1.19

3 Due by Tuesday March 31 2009

1. Prove that for asymmetric random walk, the origin is transient.
2. Prove that any two communicating states of a Markov chain are either both transient, or both recurrent.
3. Exercise 2.2
4. Exercise 2.3
5. Exercise 2.7
6. Exercise 2.8
7. Exercise 2.9
8. Exercise 2.11

   In next 2 problems we consider a branching process (X_n) with $X_0 = 1$ and branching distribution $(P_j)_{j=0}^{\infty}$.

9. Let $P_0 = P_2 = 1/2$. Find $P_{ij} = \mathsf{P}\{X_{n+1} = j |\ X_n = i\}$ for all i, j.
10. Let $G_n(s) = \mathsf{E}s^{X_n}$ and $G(s) = G_1(s) = \mathsf{E}s^{X_1}$. Prove that

$$G_n(s) = \underbrace{G(G(\ldots G}_{n}(s)\ldots))$$

11. Exercise 4.1
12. Exercise 4.2
13. Exercise 4.3
14. Exercise 4.4
15. Exercise 4.5

4 Due by April 28

Problems 3.1, 3.2, 3.3, 3.5, 3.6, 3.8(a,b), 3.9(a,b), 3.12, 3.13, 3.14.