complement[SELECT] is a set <=> axch

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summary

It is shown in this notebook that the set version of the axiom of choice is equivalent to the statement that the class of sets without cross-sections is a set. If the axiom of choice holds, then there are no sets without cross-sections, and the empty set is of course a set. If the axiom of choice fails, then there must be lots of relations without cross-sections, because then there is a relation x without a cross-section, and this implies that every set for which x is a restriction also fails to have a cross-section. That is, the class SELECT of sets with cross-sections is invariant under RESTRICT. Perhaps it would be better to look at it this way: the class complement[SELECT] is invariant under inverse[RESTRICT]. From this it follows that when axch fails, complement[SELECT] is a proper class. That is, axch is equivalent to the statement that the complement[SELECT] is a set.

the easy direction

The easy half of the theorem is that axch implies complement[SELECT] is a set, namely the empty set:

In[2]:= SubstTest[implies, equal[0, x], member[x, V], x -> complement[SELECT]]

Out[2]= or[member[complement[SELECT], V], not[axch]] = True
the reverse implication

The main idea is to use the fact that `complement[SELECT]` is invariant under `inverse[-RESTRICT]`:

\[
\text{In [4]} := \text{SubstTest}[\text{implies, and[subclass[u, v], member[v, V]], member[u, V],}
\{u \rightarrow \text{image[inverse[RESTRICT], complement}[SELECT]], v \rightarrow \text{complement}[SELECT]\}]
\]

\[
\text{Out [4]} := \text{or[member[complement}[lb]\text{image[inverse[COMPOSE], SELECT], P[Id]], V],}
\text{not[member[complement][SELECT], V]]] := \text{True}
\]

\[
\text{In [5]} := % /. \text{Equal} \rightarrow \text{SetDelayed}
\]

The final step is to use the fact that `domain[VERTSECT[\text{inverse[\text{RESTRICT}]}, V]]` is `complement[P[\text{cart}[V, V]]]`.

\[
\text{In [6]} := \text{Map[implies[member[complement][SELECT], V], \#] \&, SubstTest[member, x,}
\text{domain[IMAGE[y]], \{x \rightarrow \text{complement}[SELECT], y \rightarrow \text{inverse[RESTRICT]\}]]}
\]

\[
\text{Out [6]} := \text{or[axch, not[member[complement][SELECT], V]]] := \text{True}
\]

\[
\text{In [7]} := \text{or[axch, not[member[complement][SELECT], V]]] := \text{True}
\]

Putting together these two implications yields this logical equivalence:

\[
\text{In [8]} := \text{equiv[member[complement][SELECT], V], axch]}
\]

\[
\text{Out [8]} := \text{True}
\]

\[
\text{In [9]} := \text{member[complement}[SELECT], V] := \text{axch}
\]