summary

If the axiom of choice holds, there is no difference between the equipollence relation \( Q \) and the equicardinality relation \( \text{inverse}[\text{CARD}] \circ \text{CARD} \). By default, \text{axch} is not assumed in the GOEDEL program, so one needs to distinguish these two relations. The subset relation \( S \) commutes with \( Q \), but when the axiom of choice is not assumed, the subset relation \( S \) need not commute with the equicardinality relation \( \text{inverse}[\text{CARD}] \circ \text{CARD} \). However, the equicardinality relation does subcommute with the subset relation. This is because the domain of the cardinality function is invariant under \text{inverse}[S].

\[
\text{In}[2]:= \text{invariant}[\text{inverse}[S], \text{domain}[\text{CARD}]]
\]
\[
\text{Out}[2]= \text{True}
\]

In other words, if a set is equipollent to an ordinal, then so is any subset.

\[
\text{In}[3]:= \text{implies}[\text{and}[\text{subclass}[x, y], \text{member}[y, \text{image}[Q, \text{OMEGA}]], \text{member}[x, \text{image}[Q, \text{OMEGA}]]]]
\]
\[
\text{Out}[3]= \text{True}
\]

The statements that \( Q \) commutes with \( S \) and with \text{inverse}[S] are formulated in the GOEDEL program in an unsymmetric fashion in that the following two rewrite rules move both \( S \) and \text{inverse}[S] from left to right:

\[
\text{In}[4]:= \text{composite}[S, Q]
\]
\[
\text{Out}[4]= \text{composite}[Q, S]
\]
For the equicardinality relation, in addition to putting \( S \) or its inverse on the left or right of \( \text{inverse}[\text{CARD}] \circ \text{CARD} \), one also has the superior option of placing it in between \( \text{inverse}[\text{CARD}] \) and \( \text{CARD} \). In this notebook precise statements about all these matters are derived, as well as various new simplification rules that hold independently of \( \text{axch} \).

**a statement equivalent to axch**

The axiom of choice equivalent to the statement that every set is equipollent to an ordinal.

\[
\text{equiv}[\text{axch}, \text{equal}[	ext{V}, \text{image}[	ext{Q}, \text{OMEGA}]]]
\]

\[\text{Out}[6]= \text{True}\]

Lemma.

\[
\text{SubstTest}[\text{implies}, \text{equal}[	ext{u}, \text{v}], \text{equal}[	ext{fix}[	ext{u}], \text{fix}[	ext{v}]],
\{\text{u} \rightarrow \text{Q}, \text{v} \rightarrow \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]}\}] \quad \text{// Reverse}
\]

\[\text{Out}[7]= \text{or}[\text{axch}, \text{not}[	ext{equal}[	ext{Q}, \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]]}]] = \text{True}\]

\[\text{In}[8]= \% /. \text{Equal} \to \text{SetDelayed}\]

Lemma. The equipollence relation and the equicardinality relation are equal if \( \text{axch} \) holds.

\[
\text{implies}[\text{axch}, \text{equal}[	ext{Q}, \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]]}] \quad \text{// AssertTest}
\]

\[\text{Out}[9]= \text{or}[	ext{equal}[	ext{Q}, \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]]}, \text{not}[\text{axch}]] = \text{True}\]

\[\text{In}[10]= \% /. \text{Equal} \to \text{SetDelayed}\]

Theorem. The statement that the equipollence and equicardinality relations are equal is equivalent to \( \text{axch} \).

\[
\text{equiv}[\text{equal}[	ext{Q}, \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]]}, \text{axch}]
\]

\[\text{Out}[11]= \text{True}\]

\[\text{In}[12]= \text{equal}[	ext{Q}, \text{composite[\text{inverse}[\text{CARD}], \text{CARD}]]} := \text{axch}\]

**a key simplification rule**

Lemma. An inclusion.

\[
\text{SubstTest}[\text{subclass}, \text{x}, \text{composite[\text{id}[\text{y}, \text{z}]],} \{\text{x} \rightarrow \text{composite[\text{inverse}[\text{CARD}], \text{S}, \text{CARD}],}
\text{y} \rightarrow \text{domain[\text{CARD}],} \text{z} \rightarrow \text{composite[\text{Q}, \text{S}]}\}] \quad \text{// Reverse}
\]

\[\text{Out}[13]= \text{subclass[composite[\text{inverse}[\text{CARD}], \text{S}, \text{CARD}], composite[\text{inverse}[\text{CARD}], \text{CARD}, \text{S}]]} = \text{True}\]
Lemma. An equation.

In[15]:= equal[composite[inverse[CARD], S, CARD],
       composite[inverse[CARD], CARD, S]] // AssertTest
Out[15]= equal[composite[inverse[CARD], CARD, S], composite[inverse[CARD], S, CARD]] = True

In[16]:= % /. Equal -> SetDelayed
A better equation can be derived.

Theorem. A simplification rule for moving CARD past S.

In[17]:= SubstTest[implies, equal[u, v], equal[composite[t, u], composite[t, v]],
  {t -> CARD, u -> composite[inverse[CARD], CARD, S],
   v -> composite[inverse[CARD], S, CARD]}; // Reverse
Out[17]= equal[composite[CARD, S], composite[id[fix[CARD]]], S, CARD]] = True

In[18]:= composite[CARD, S] := composite[id[fix[CARD]]], S, CARD]
Corollary.

In[19]:= composite[inverse[S], inverse[CARD]] // DoubleInverse
Out[19]= composite[inverse[S], inverse[CARD]] =
       composite[inverse[CARD], inverse[S], id[fix[CARD]]]

In[20]:= composite[inverse[S], inverse[CARD]] :=
       composite[inverse[CARD], inverse[S], id[fix[CARD]]]

Theorem. A rewrite rule that moves the inverse of the cardinality function to the left.

In[21]:= Map[composite[#, inverse[CARD]] &,
  Assoc[composite[id[image[Q, OMEGA]]], S, id[domain[CARD]]], Q]]
Out[21]= composite[id[image[Q, OMEGA]]], S, inverse[CARD]] =
       composite[inverse[CARD], S, id[fix[CARD]]]

In[22]:= composite[id[image[Q, OMEGA]]], S, inverse[CARD]] :=
       composite[inverse[CARD], S, id[fix[CARD]]]
Corollary. A rewrite rule that moves the cardinality function to the right.

In[26]:= composite[CARD, inverse[S], id[image[Q, OMEGA]]] // DoubleInverse
Out[26]= composite[CARD, inverse[S], id[image[Q, OMEGA]]] =
       composite[id[fix[CARD]]], inverse[S], CARD]

In[27]:= composite[CARD, inverse[S], id[image[Q, OMEGA]]] :=
       composite[id[fix[CARD]]], inverse[S], CARD]
Corollary. This special result is not needed in the rest of this notebook.

In[29]:= \text{Map[composite[\text{CARD}, \#] \\ &,
Assoc[id[\text{FINITE}], id[\text{image[Q, OMEGA]}], composite[S, inverse[\text{CARD}]]]}

Out[29]= \text{composite[\text{id[omega], S, id[fix[\text{CARD}]]]} = \text{composite[\text{id[omega], S, id[omega]]}}

In[30]:= composite[\text{id[omega], S, id[fix[\text{CARD}]]]} := \text{composite[\text{id[omega], S, id[omega]]}}

---

\text{subcommute property}

Corollary. The equipollence relation subcommutes with the subset relation.

In[31]:= \text{SubstTestsubclass, composite[id[x], y, y,
\{x \rightarrow \text{domain[\text{CARD}], y \rightarrow composite[S, inverse[\text{CARD}], \text{CARD}]}\} // \text{Reverse}}

Out[31]= \text{subclass[composite[\text{inverse[\text{CARD}], S, \text{CARD}], composite[S, inverse[\text{CARD}], \text{CARD}}]} = \text{True}

In[32]:= \text{subclass[composite[\text{inverse[\text{CARD}], S, \text{CARD}], composite[S, inverse[\text{CARD}], \text{CARD}}]}} := \text{True}

Restatement.

In[33]:= \text{subcommute[composite[inverse[\text{CARD}], \text{CARD}], S]}

Out[33]= \text{True}

Corollary. The relation \text{inverse[S]} subcommutes with the equicardinality relation.

In[34]:= \text{SubstTest[\text{subcommute, inverse[x], inverse[y],}
\{x \rightarrow \text{composite[\text{inverse[\text{CARD}], CARD}], y \rightarrow \text{inverse[S]}\}]}

Out[34]= \text{subclass[composite[\text{inverse[\text{CARD}], inverse[S], \text{CARD}], composite[\text{inverse[\text{CARD}], CARD, inverse[S]}]]} = \text{True}

In[35]:= \text{subclass[composite[\text{inverse[\text{CARD}], inverse[S], \text{CARD}],}
\text{composite[\text{inverse[\text{CARD}], CARD, inverse[S]}]]}} := \text{True}

Restatement.

In[36]:= \text{subcommute[\text{inverse[S], composite[\text{inverse[\text{CARD}], \text{CARD}]}}]}

Out[36]= \text{True}

---

\text{another statement equivalent to axch}

Lemma. If \text{axch} holds, then the equicardinality relation commutes with the subset relation.
Lemma. If the equicardinality relation commutes with the subset relation, then \texttt{axch} holds.

Theorem. The statement that the equicardinality relation commutes with the subset relation is equivalent to \texttt{axch}.

Restatement.

**rules that convert \texttt{Q} to \texttt{inverse[CARD]} \circ \texttt{CARD}**

By adding a factor of \texttt{id[image[Q, \Omega]]}, one can use double inversion to convert an expression involving the equipollence relation to one involving the equicardinality relation.
In[46]:= composite[Q, inverse[S], id[image[Q, OMEGA]]] // DoubleInverse

Out[46]= composite[Q, inverse[S], id[image[Q, OMEGA]]] :=
    composite[inverse[CARD], inverse[S], CARD]

In[47]:= composite[Q, inverse[S], id[image[Q, OMEGA]]] :=
    composite[inverse[CARD], inverse[S], CARD]

---

**rules that remove a factor of S or its inverse**

Theorem.

In[48]:= Assoc[composite[Q, S], id[image[Q, OMEGA]], S] // Reverse


In[49]:= composite[S, inverse[CARD], S, CARD] := composite[S, inverse[CARD], CARD]

Corollary.

In[50]:= composite[inverse[CARD], inverse[S], CARD, inverse[S]] // DoubleInverse

Out[50]= composite[inverse[CARD], inverse[S], CARD, inverse[S]] :=
    composite[inverse[CARD], CARD, inverse[S]]

In[51]:= composite[inverse[CARD], inverse[S], CARD, inverse[S]] :=
    composite[inverse[CARD], CARD, inverse[S]]

---

**rules that remove a factor of CARD or its inverse**

Theorem.

In[52]:= Map[inverse, Assoc[inverse[CARD], composite[Q, S], inverse[CARD]]]

Out[52]= composite[CARD, inverse[S], CARD] := composite[id[fix[CARD]], inverse[S], CARD]

In[53]:= composite[CARD, inverse[S], CARD] := composite[id[fix[CARD]], inverse[S], CARD]

Corollary.

In[54]:= composite[inverse[CARD], S, inverse[CARD]] // DoubleInverse

Out[54]= composite[inverse[CARD], S, inverse[CARD]] := composite[inverse[CARD], S, id[fix[CARD]]]

In[55]:= composite[inverse[CARD], S, inverse[CARD]] :=
    composite[inverse[CARD], S, id[fix[CARD]]]
rules that remove an identity factor

Theorem.

\[
\text{In}[56]:= \text{Assoc\{inverse[CARD], composite\{id\{image\{Q, OMEGA\}\}, S\}, id\{image\{Q, OMEGA\}\}\} // Reverse}
\]

\[
\text{Out}[56]= \text{composite\{inverse[CARD], S, id\{image\{Q, OMEGA\}\}\} = composite\{inverse[CARD], S\}}
\]

\[
\text{In}[57]:= \text{composite\{inverse[CARD], S, id\{image\{Q, OMEGA\}\}\} := composite\{inverse[CARD], S\}}
\]

Corollary.

\[
\text{In}[58]:= \text{composite\{id\{image\{Q, OMEGA\}\}, inverse[S], CARD\} // DoubleInverse}
\]

\[
\text{Out}[58]= \text{composite\{id\{image\{Q, OMEGA\}\}, inverse[S], CARD\} = composite\{inverse[S], CARD\}}
\]

\[
\text{In}[59]:= \text{composite\{id\{image\{Q, OMEGA\}\}, inverse[S], CARD\} := composite\{inverse[S], CARD\}}
\]