wellordering theorem

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In[1]:= SetDirectory["1:”]; << goedel.12april12a

:Package Title: goedel.12april12a 2012 April 12 at 1:15 p.m.
Loading takes about seventeen minutes, half that time due to builtin pauses.
It is now: 2012 Apr 12 at 14:52
Loading Simplification Rules
TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3
weightlimit = 40
Loading completed.
It is now: 2012 Apr 12 at 15:9

summary

A set $x$ is well-orderable if there is a well-ordering $w$ with $x = \text{fix}[w]$. In this notebook it is shown that if a set can be well-ordered, then it is equipollent to an ordinal. The converse is also true. It follows that a set can be well-ordered if and only if it is equipollent to an ordinal. This is true independently of the axiom of choice. The proof uses well-founded recursion, and the fact that a set is equipollent to an ordinal if and only if it is equipollent to a set of ordinals.

In[2]:= image[Q, P[OMEGA]]

Out[2]= image[Q, OMEGA]

The standard formulation of the well-ordering theorem follows as a corollary, that axch is equivalent to the statement than any set can be well-ordered. The following slightly weaker form of this is already available in the GOEDEL program.

In[3]:= equal[V, image[Q, OMEGA]]

Out[3]= axch

This says that axch is equivalent to the statement that any set is equipollent to an ordinal.

strict monotonicity for well-orderings

Since any well-order is a total order, strictly monotone functions are bijections.
Technical Lemma. One can add a harmless factor \(\text{id}[y]\).

Explicit reference to the recursion construction can be eliminated. All one needs to know is the existence of a function with the above property.

Theorem. There is a strictly monotone function from the range of any small well-founded relation to a set of ordinals.
Lemma. An inclusion.

In[11]:= SubstTest[implies, subclass[u, v], subclass[monotone[x, u], monotone[x, v]],
{u \rightarrow \text{composite[id[OMEGA], E], v \rightarrow Di}}] // Reverse


In[12]:= subclass[monotone[x_, \text{composite[id[OMEGA], E]}], monotone[x_, Di]] := True

Lemma. Another version of the statement that there is a strictly monotone mapping from the range of a small well-founded relation to the class of ordinals.

In[13]:= Map[not, SubstTest[implies, member[u, v], not[empty[v]],
{u \rightarrow \text{composite[rec[composite[TC, IMAGE[SECOND], SECOND], inverse[\text{wf[setpart[x]]}]]], id[range[\text{wf[setpart[x]]}]]}, v \rightarrow \text{intersection[map[range[\text{wf[setpart[x]]}], OMEGA], \text{monotone[\text{wf[setpart[x]]}, \text{composite[id[OMEGA], E]}]]}]}] // Reverse

Out[13]= equal[0, \text{intersection[map[range[\text{wf[setpart[x]]}], OMEGA], \text{monotone[\text{wf[setpart[x]]}, \text{composite[id[OMEGA], E]}]]}}] := False

In[14]:= equal[0, \text{intersection[map[range[\text{wf[setpart[x_]}], OMEGA], \text{monotone[\text{wf[setpart[x_]}, \text{composite[id[OMEGA], E]}]]}]}] := False

Theorem. Yet another statement about the existence of a strictly monotone function, using the diversity relation Di.

In[15]:= SubstTest[and, empty[v], subclass[u, v],
{u \rightarrow \text{intersection[map[range[\text{wf[setpart[x]]}], OMEGA], \text{monotone[\text{wf[setpart[x]]}, \text{composite[id[OMEGA], E]}]], v \rightarrow \text{intersection[map[range[\text{wf[setpart[x]]}], OMEGA], \text{monotone[\text{wf[setpart[x]}, Di]]}]}]}] // Reverse

Out[15]= equal[0, \text{intersection[map[range[\text{wf[setpart[x]]}], OMEGA], \text{monotone[\text{wf[setpart[x]}, Di]}]]}] := False

In[16]:= equal[0, \text{intersection[map[range[\text{wf[setpart[x_]}], OMEGA], \text{monotone[\text{wf[setpart[x_]}, Di]}]}]] := False

The strict part of any well-order is well-founded. The following result follows from the above theorem.

Corollary. Existence of strictly monotone mappings to $\Omega$ for well-orders.

In[17]:= SubstTest[equal, 0, \text{intersection[map[range[\text{wf[setpart[t]]}], OMEGA], \text{monotone[\text{wf[setpart[t]}, Di]], t \rightarrow \text{intersection[wo[setpart[x]], Di]}]]} // Reverse

Out[17]= equal[0, \text{intersection[map[fix[\text{composite[wo[setpart[x]], Di]]}, OMEGA], \text{monotone[\text{intersection[Di, wo[setpart[x]], Di]}}}]]] := False

In[18]:= equal[0, \text{intersection[map[fix[\text{composite[wo[setpart[\_]}, Di]]}, OMEGA], \text{monotone[\text{intersection[Di, wo[setpart[\_]]], Di]}]}]] := False

Lemma. A simplification rule.
Theorem. There is one-to-one mapping from the range of the strict part of a well-order to a set of ordinals.

Corollary. The range of the strict part of a small well-order is equipollent to an ordinal.

Lemma. The function part of a total order is finite.
Theorem. The fixed point set of a (small) well-order is equipollent to an ordinal.

Theorem. A similar result for a well-ordering.

Theorem. A similar result for the inverse of a total order.
Lemma. A simplification rule.

\[
\text{In[42]:=} \quad \text{Assoc[IMAGE[inverse[DUP]],}
\text{IMAGE[id[cart[V, V]]], inverse[IMAGE[id[cart[V, V]]]]]} /\text{ Reverse}
\]

\[
\text{Out[42]= composite[IMAGE[inverse[DUP]], inverse[IMAGE[id[cart[V, V]]]]] =}
\text{composite[IMAGE[inverse[DUP]], id[P[cart[V, V]]]]}
\]

\[
\text{In[43]:=} \quad \text{composite[IMAGE[inverse[DUP]], inverse[IMAGE[id[cart[V, V]]]]]} :=
\text{composite[IMAGE[inverse[DUP]], id[P[cart[V, V]]]]}
\]

Lemma. A simplification rule.

\[
\text{In[44]:=} \quad \text{ImageComp[IMAGE[inverse[DUP]], inverse[IMAGE[id[cart[V, V]]]], x]} /\text{ Reverse}
\]

\[
\text{Out[44]= image[IMAGE[inverse[DUP]], image[inverse[IMAGE[id[cart[V, V]]]], x]] =}
\text{image[IMAGE[inverse[DUP]], intersection[x, P[cart[V, V]]]]}
\]

\[
\text{In[45]:=} \quad \text{image[IMAGE[inverse[DUP]], image[inverse[IMAGE[id[cart[V, V]]]], x_]] :=}
\text{image[IMAGE[inverse[DUP]], intersection[x, P[cart[V, V]]]]}
\]

A variable-free statement is derived using \text{reify} and \text{case}.

Lemma. The class of fixed point sets of well-orderings is a subclass of the class of sets equipollent to an ordinal.

\[
\text{In[46]:=} \quad \text{Map[equal[domain[#, V] &,}
\text{SubstTest[reify, x, case{member[fix[wo[setpart[x]]], y]}, y \rightarrow \text{image[Q, OMEGA]]]}]
\]

\[
\text{Out[46]= subclass[image[IMAGE[inverse[DUP]], WO], \text{image[Q, OMEGA]}]} = \text{True}
\]

\[
\text{In[47]:=} \quad \% /\text{. Equal \rightarrow SetDelayed}
\]

Main Theorem. The class of fixed point sets of well-orderings is equal to the class of sets equipollent to an ordinal.

\[
\text{In[48]:=} \quad \text{SubstTest[and, subclass[u, v], subclass[v, u],}
\{u \rightarrow \text{image[IMAGE[inverse[DUP]], WO], v} \rightarrow \text{image[Q, OMEGA]}\}]
\]

\[
\text{Out[48]= equal[image[Q, OMEGA], \text{image[IMAGE[inverse[DUP]], WO]}]} = \text{True}
\]

\[
\text{In[49]:=} \quad \text{image[IMAGE[inverse[DUP]], WO] := image[Q, OMEGA]}
\]

It is to be emphasized that everything up to here has been derived without using the axiom of choice.

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### the well-ordering theorem

The usual statement of the well-ordering theorem follows as a corollary of the main theorem derived in the preceding section. The axiom of choice is equivalent to the statement that every set can be well-ordered. No new rewrite rule is needed here.
In[50]:= equal[V, image[IMAGE[inverse[DUP]], WO]]

Out[50]= axch