max-min monus formulas

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In [1] := SetDirectory["i:"]; << goedel70.30a; << tools.m

:Package Title: goedel70.30a 2005 June 30 at 9:00 a.m.

It is now: 2005 Jul 2 at 18:1

Loading Simplification Rules

TOOLS.M Revised 2005 June 17

weightlimit = 40

summary

In Art Quaife's development of the arithmetic of natural numbers, using William McCune's automated reasoning program Otter, extensive use is made of floored subtraction.


Comparing Quaife's formalism with that of the GOEDEL program is facilitated by interpreting floored subtraction as monus[x,y], defined by wrapping nat around natsub[x,y].

In [3] := monus[x, y]

Out [3] = nat[natsub[x, y]]

If x and y are natural numbers, then monus[x,y] is their difference when y is less than or equal to x, and is zero otherwise. Two formulas for monus are derived in this notebook which involve the minimum and maximum of two natural numbers. In Gödel's formalism, each natural number is the set of all lesser ones, and therefore the minimum and maximum of two natural numbers are simply their intersection and union, respectively. Quaife's definitions of the maximum and minimum of two numbers now become theorems, derived here as corollaries.
two formulas for monus

Lemma 1.

\[
\text{In[4]:= } \text{SubstTest[and, subclass[u, v], subclass[v, u],}\\
\{u \to \text{natsub[nat[x], nat[y]], } v \to \text{natsub[nat[x], nat[z]]}\} / .\\
z \to \text{intersection[nat[x], nat[y]]} // \text{Reverse}
\]

\[
\text{Out[4]= } \text{equal[natsub[nat[x], intersection[nat[x], nat[y]]], natsub[nat[x], nat[y]]]} =\\
\text{not[member[nat[x], nat[y]]]}
\]

\[
\text{In[5]:= } \text{equal[natsub[nat[x_], intersection[nat[x_], nat[y_]]],}\\
\text{natsub[nat[x_], nat[y_]]} = : \text{not[member[nat[x], nat[y]]]}
\]

Lemma 2.

\[
\text{In[6]:= } \text{SubstTest[implies, and[equal[u, v], equal[v, w]],}\\
equal[u, w], \{u \to \text{natsub[nat[x], intersection[nat[x], nat[y]]],}\\v \to \text{natsub[nat[x], nat[y]], } w \to \text{monus[nat[x], nat[y]]}\}\\
\text{Out[6]= } \text{or[equal[nat[natsub[nat[x], nat[y]]],}\\
\text{natsub[nat[x], intersection[nat[x], nat[y]]]],}\\\text{member[nat[x], nat[y]]} = \text{True}
\]

\[
\text{In[7]:= } \{u \to x_, } v \to y_\} / . \text{Equal} \to \text{SetDelayed}
\]

Lemma 3.

\[
\text{In[8]:= } \text{SubstTest[implies, and[equal[u, v], equal[v, w]],}\\
equal[u, w], \{u \to \text{natsub[nat[x], intersection[nat[x], nat[y]]],}\\v \to 0, } w \to \text{monus[nat[x], nat[y]]}\}\\
\text{Out[8]= } \text{or[equal[nat[natsub[nat[x], nat[y]]],}\\
\text{natsub[nat[x], intersection[nat[x], nat[y]]]],}\\\text{member[nat[y], nat[x]]} = \text{True}
\]

\[
\text{In[9]:= } \{u \to x_, } v \to y_\} / . \text{Equal} \to \text{SetDelayed}
\]

The first of two formulas for \textbf{monus} now follows, and is made into a rewrite rule.

\[
\text{In[10]:= } \text{SubstTest[and, or[p1, p3], or[p2, p3],}\\
\{p1 \to \text{member[nat[x], nat[y]], } p2 \to \text{member[nat[y], nat[x]], } p3 \to \text{equal[nat[natsub[nat[x], nat[y]]],}\\\text{natsub[nat[x], intersection[nat[x], nat[y]]]}\}] // \text{Reverse}
\]

\[
\text{Out[10]= } \text{equal[nat[natsub[nat[x], nat[y]]],}\\\text{natsub[nat[x], intersection[nat[x], nat[y]]]} = \text{True}
\]
The second formula for \textit{monus} is obtained as a corollary, and also made into a rewrite rule.

In a previous notebook, dealing with Quaife's DF group, the following formula was derived for expressions with \textit{monus} substituted inside \texttt{natadd}.

This complicated rewrite rule is removed here, and will be replaced with a simpler rewrite rule that corresponds to Quaife's second equation of his (MMDEF1).

This equation, which was used by Quaife to define the maximum of two natural numbers, suffices for the \texttt{GODEL} program to recognize the validity of all theorems of Quaife's DF group. It can be quickly derived as follows:

In a similar fashion, the first equation of Quaife's (MMDEF1), which he used to define the minimum of two numbers, can be derived from the first formula for \textit{monus} by double transposition:
\[ \text{In[18]} := \text{SubstTest[implies, equal[natsub[nat[u], nat[v]], nat[w]],}
\]
\[ \text{equal[natsub[nat[u], nat[w]], nat[v]},}
\]
\[ \{u \rightarrow \text{natsub[nat[x], nat[y]], v \rightarrow \text{intersection[nat[x], nat[y]], w \rightarrow \text{natsub[nat[x], nat[y]]}}\}] \]

\[ \text{Out[18]} = \text{equal[intersection[nat[x], nat[y]],}
\]
\[ \text{natsub[nat[x], nat[natsub[nat[x], nat[y]]]]] = \text{True} \]

\[ \text{In[19]} := \text{natsub[nat[x_], natsub[nat[x_], nat[y_]]]] := intersection[nat[x], nat[y]]} \]