The Smullyan-Fitting double induction theorem is applied to the set of finite levels of the Zermelo-von Neumann cumulative hierarchy.

**summary**

The Smullyan-Fitting double induction theorem is applied to the set of finite levels of the Zermelo-von Neumann cumulative hierarchy.

**introduction: the power tower**

The finite levels of the Zermelo-von Neumann cumulative hierarchy are the sets \(0, P[0], P[P[0]], \ldots\) obtained by iterating the power set functor, starting with the empty set. These are the values of the function \(\text{IMAGE}[\text{inverse}[\text{RANK}]]\) at the natural numbers:

\[
\text{In}[2]:= \text{APPLY}[\text{IMAGE}[\text{inverse}[\text{RANK}]], 0] \\
\text{Out}[2]= 0 \\
\text{In}[3]:= \text{APPLY}[\text{IMAGE}[\text{inverse}[\text{RANK}]], \text{set}[0]] = P[0] \\
\text{Out}[3]= \text{True} \\
\text{In}[4]:= \text{APPLY}[\text{IMAGE}[\text{inverse}[\text{RANK}]], \text{succ}[\text{set}[0]]] = P[P[0]] \\
\text{Out}[4]= \text{True} \\
\text{In}[5]:= \text{APPLY}[\text{IMAGE}[\text{inverse}[\text{RANK}]], \text{succ}[\text{succ}[\text{set}[0]]]] = P[P[P[0]]] \\
\text{Out}[5]= \text{True}
\]

Note that these sets form an infinite chain; each one is a subset of the next one, obtained by iteratively applying the function \(\text{POWER}\). The set of all these finite levels of the cumulative hierarchy can be described as the orbit of \(\text{POWER}\) starting at \(0\), that is, the minimal set which holds the empty set and is invariant under \(\text{POWER}\).
In this notebook, some rewrite rules related to this collection of sets are derived, and an application is made of the corollary about progressive functions of the double induction theorem presented by Smullyan and Fitting. See the notebooks dbl-ind1.nb and dbl-ind2.nb for details.

The function to which this corollary is applied is not POWER itself, but its restriction to the class of full sets.

**lemmas**

**Lemma.**

\[
\text{subclass}[\text{range}[\text{iterate}[\text{POWER}, \text{set}[0]]], \text{omega}] 
\]

**Corollary.**

\[
\text{equal}[\text{intersection}[\text{omega}, \text{image}[\text{inverse}[\text{IMAGE}[\text{inverse}[\text{RANK}]]], \text{FULL}]], \text{omega}] 
\]

**Lemma.**

\[
\text{Assoc}[\text{IMAGE}[\text{inverse}[\text{RANK}]], \text{composite}[\text{id}[\text{OMEGA}], \text{SUCCESS}], \text{id}[\text{omega}]] 
\]

An application of the uniqueness of iteration.
The range of iterate is a minimal invariant class.

**Corollary.**

**Lemma.**

**Corollary.**
Restatement: The levels of the power tower are totally ordered by inclusion.

Restatement: The levels of the power tower are totally ordered by inclusion.