PO and EQV

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summary

Some variable-free rewrite rules for equivalences and partial orderings are derived. Most of the rules for PO were initially discovered by using reify to remove wrapped variables. Quite a few simplification rules were needed to accomplish this, presumably due to the complexity of the definition of the po wrapper. Once the formulas were known, however, it was found to be relatively easy to obtain direct derivations of these results without resorting to reification. The EQV rules are derived in a completely analogous fashion.

RFX rules

Lemma.

\[\text{symdif[composite[IMAGE[FIRST], id[RFX]], composite[IMAGE[inverse[DUP]], id[RFX]]] // RelnRenormality}\]

Theorem.

\[\text{SubstTest[equal, 0, symdif[u, v], }\{u \to \text{composite[IMAGE[FIRST], id[RFX]]}, v \to \text{composite[IMAGE[inverse[DUP]], id[RFX]]}\}]\]

\[\text{true = equal[composite[IMAGE[FIRST], id[RFX]], composite[IMAGE[inverse[DUP]], id[RFX]]]}\]

\[\text{composite[IMAGE[FIRST], id[RFX]] := composite[IMAGE[inverse[DUP]], id[RFX]]}\]
Corollary.

\[ \text{In [6]} := \text{Map[composite[#, id[RFX]] \&, Assoc[IMAGE[FIRST], id[RFX], IMAGE[SWAP]]]} \]

\[ \text{Out [6]} = \text{composite[IMAGE[SECOND], id[RFX]]} = \text{composite[IMAGE[inverse[DUP]], id[RFX]]} \]

\[ \text{In [7]} := \text{composite[IMAGE[SECOND], id[RFX]]} := \text{composite[IMAGE[inverse[DUP]], id[RFX]]} \]

The rules in this section state that for a reflexive relation, the domain, range and fixed point sets are all equal.

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**ANTI SYM rule**

Lemma.

\[ \text{In [8]} := \text{symdif[composite[CORE[SYM], id[ANTISYM]],}
\]
\[ \quad \text{composite[IMAGE[id[Id]], id[ANTISYM]]}] \text{ // domain // Normality} \]

\[ \text{Out [8]} = \text{union[intersection[ANTISYM,}
\]
\[ \quad \text{complement[fix[composite[inverse[CORE[SYM]], IMAGE[id[Id]]]]],}
\]
\[ \quad \text{intersection[}
\]
\[ \quad \text{ANTISYM, complement[fix[composite[inverse[IMAGE[id[Id]], CORE[SYM]]]]]]} = 0 \]

\[ \text{In [9]} := \% . \text{Equal \rightarrow SetDelayed} \]

Theorem.

\[ \text{In [10]} := \text{SubstTest[equal, 0, domain[symdif[u, v]],}
\]
\[ \quad \{u \rightarrow \text{composite[CORE[SYM], id[ANTISYM]],} v \rightarrow \text{composite[IMAGE[id[Id]], id[ANTISYM]]}\}] \]

\[ \text{Out [10]} = \text{True} = \text{equal[composite[CORE[SYM], id[ANTISYM]],}
\]
\[ \quad \text{composite[IMAGE[id[Id]], id[ANTISYM]]} \]

\[ \text{In [11]} := \text{composite[CORE[SYM], id[ANTISYM]]} := \text{composite[IMAGE[id[Id]], id[ANTISYM]]} \]

The function CORE[SYM] yields the largest symmetric relation contained in a given relation. This is obtained by simply intersecting a relation with its inverse.

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**IDEM rule**

A relation \( x \) is idempotent if \( \text{composite}[x, x] \) is equal to \( x \).

\[ \text{In [12]} := \text{SubstTest[composite, funpart[x], id[fix[funpart[x]]], x \rightarrow \text{composite[COMPOSE, DUP]}]} \]

\[ \text{Out [12]} = \text{composite[COMPOSE, DUP, id[IDEM]]} = \text{id[IDEM]} \]

\[ \text{In [13]} := \text{composite[COMPOSE, DUP, id[IDEM]]} := \text{id[IDEM]} \]
**EQV rules**

Equivalence relations are reflexive, symmetric and transitive relations. The rules for EQV are immediate corollaries of the rules derived above.

```plaintext
In[14]:= Assoc[IMAGE[id[cart[V, V]]], id[cart[V, V]]], id[EQV]]
Out[14]= composite[IMAGE[id[cart[V, V]]], id[EQV]] = id[EQV]

In[15]:= composite[IMAGE[id[cart[V, V]]], id[EQV]] := id[EQV]
```

From the reflexive property one finds:

```plaintext
In[16]:= Assoc[IMAGE[FIRST], id[RFX], id[EQV]]
Out[16]= composite[IMAGE[FIRST], id[EQV]] = composite[IMAGE[inverse[DUP]], id[EQV]]

In[17]:= composite[IMAGE[FIRST], id[EQV]] := composite[IMAGE[inverse[DUP]], id[EQV]]

In[18]:= Assoc[IMAGE[SECOND], id[RFX], id[EQV]]
Out[18]= composite[IMAGE[SECOND], id[EQV]] = composite[IMAGE[inverse[DUP]], id[EQV]]

In[19]:= composite[IMAGE[SECOND], id[EQV]] := composite[IMAGE[inverse[DUP]], id[EQV]]
```

From the symmetric property one finds:

```plaintext
In[20]:= Assoc[CORE[SYM], id[SYM], id[EQV]]
Out[20]= composite[CORE[SYM], id[EQV]] = id[EQV]

In[21]:= composite[CORE[SYM], id[EQV]] := id[EQV]
```

The reflexive and transitive properties imply idempotence, from which one finds:

```plaintext
In[22]:= Assoc[composite[COMPOSE, DUP], id[IDEM], id[EQV]]
Out[22]= composite[COMPOSE, DUP, id[EQV]] = id[EQV]

In[23]:= composite[COMPOSE, DUP, id[EQV]] := id[EQV]
```

**EQUIVALENCE rule**

```plaintext
In[24]:= equiv[and[EQUIVALENCE[x], REFLEXIVE[x]], EQUIVALENCE[x]]
Out[24]= True

In[25]:= and[EQUIVALENCE[x__], REFLEXIVE[x__]] := EQUIVALENCE[x]
```
# PO rules

Partial orderings are reflexive, antisymmetric and transitive relations. The rules for PO are also immediate corollaries of rules derived above.

\[
\text{In [26]} := \text{Assoc[\text{IMAGE[id[cart[V, V]]], id[cart[V, V]], id[PO]]}}
\]
\[
\text{Out[26]} = \text{composite[\text{IMAGE[id[cart[V, V]]], id[PO]]] := id[PO]}
\]

\[
\text{In[27]} := \text{composite[\text{IMAGE[id[cart[V, V]]], id[PO]]] := id[PO]}
\]

From the reflexive property:

\[
\text{In [28]} := \text{Assoc[\text{IMAGE[FIRST], id[RFX], id[PO]]}}
\]
\[
\text{Out[28]} = \text{composite[\text{IMAGE[FIRST], id[PO]]} \Rightarrow \text{composite[\text{IMAGE[inverse[DUP]], id[PO]]}}
\]

\[
\text{In[29]} := \text{composite[\text{IMAGE[FIRST], id[PO]]} := \text{composite[\text{IMAGE[inverse[DUP]], id[PO]]}}
\]

\[
\text{In [30]} := \text{Assoc[\text{IMAGE(SECOND), id[RFX], id[PO]]}}
\]
\[
\text{Out[30]} = \text{composite[\text{IMAGE(SECOND), id[PO]]} \Rightarrow \text{composite[\text{IMAGE[inverse[DUP]], id[PO]]}}
\]

\[
\text{In[31]} := \text{composite[\text{IMAGE(SECOND), id[PO]]} := \text{composite[\text{IMAGE[inverse[DUP]], id[PO]]}}
\]

The antisymmetric property:

\[
\text{In [32]} := \text{Assoc[\text{CORE[SYM], id[ANTISYM], id[PO]]}}
\]
\[
\text{Out[32]} = \text{composite[\text{CORE[SYM], id[PO]]} \Rightarrow \text{composite[\text{IMAGE[id[Id]], id[PO]]}}
\]

\[
\text{In[33]} := \text{composite[\text{CORE[SYM], id[PO]]} := \text{composite[\text{IMAGE[id[Id]], id[PO]]}}
\]

The idempotence property:

\[
\text{In [34]} := \text{Assoc[\text{composite[COMPOSE, DUP], id[IDEM], id[PO]]}}
\]
\[
\text{Out[34]} = \text{composite[\text{COMPOSE, DUP, id[PO]]} \Rightarrow \text{id[PO]}}
\]

\[
\text{In[35]} := \text{composite[\text{COMPOSE, DUP, id[PO]]} := \text{id[PO]}}
\]