finite T1 spaces are discrete

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\texttt{In[1]}:= << goedel54.07b; << tools.m

:Package Title: goedel54.07b 2004 February 7 at 4:25 p.m.

It is now: 2004 Feb 9 at 13:44

Loading Simplification Rules

TOOLS.M Revised 2004 January 3

weightlimit = 40

\textbf{summary}

Any T1 topology on a finite space is discrete.

\textbf{any PointClosed family of finite sets is finite}

Lemma: Finiteness is unaffected by adding or deleting a single element.

\texttt{In[2]}:= SubstTest[member, union[u, v], FINITE, 
\{u \rightarrow \text{singleton}[y], v \rightarrow \text{dif}[x, \text{singleton}[y]]\}] \quad \text{// Reverse}

\texttt{Out[2]}= member[\text{intersection}[x, \text{complement}[\text{singleton}[y]]], \text{FINITE}] = member[x, \text{FINITE}]

\texttt{In[3]}:= member[\text{intersection}[x_\_, \text{complement}[\text{singleton}[y_\_]], \text{FINITE}] := member[x, \text{FINITE}]

Lemma. A collection \( t \) of finite sets which holds the relative complement in \( U[t] \) of some singleton must be a finite collection.

\texttt{In[4]}:= SubstTest[\text{implies}, \text{and}[\text{member}[u, t], \text{subclass}[t, w]], \text{member}[u, w],
\{u \rightarrow \text{dif}[U[t], \text{singleton}[x]], w \rightarrow \text{FINITE}\}]

\texttt{Out[4]}= \text{or}[\text{member}[t, \text{FINITE}], \text{not}[\text{member}[\text{intersection}[\text{complement}[\text{singleton}[x]], U[t]], t]],
\text{not}[\text{subclass}[t, \text{FINITE}]]= \text{True}]

\texttt{In[5]}= \% /. \{x \rightarrow x_\_, t \rightarrow t\_\} \quad \text{// Equal} \rightarrow \text{SetDelayed}

If \( t \) is a point–closed collection of finite sets, and if there is a point \( x \) in \( U[t] \), then \( t \) is finite.

\texttt{In[6]}:= Map[\text{not}, \text{SubstTest}[\text{and}, \text{implies}[\text{and}[p1, p2], p4],
\text{implies}[\text{and}[p3, p4], p5], \text{not}[\text{implies}[\text{and}[p1, p2, p3, p5]],
\{p1 \rightarrow \text{member}[x, U[t]], p2 \rightarrow \text{member}[t, \text{PointClosed}], p3 \rightarrow \text{subclass}[t, \text{FINITE}],
p4 \rightarrow \text{member}[\text{dif}[U[t], \text{singleton}[x]], t]], p5 \rightarrow \text{member}[t, \text{FINITE}]\}]]

\texttt{Out[6]}= \text{or}[\text{member}[t, \text{FINITE}], \text{not}[\text{member}[t, \text{PointClosed}],
\text{not}[\text{member}[x, U[t]]], \text{not}[\text{subclass}[t, \text{FINITE}]]= \text{True}]}
In[7]:= (% /. {t -> t_}) /. Equal -> SetDelayed

There are only two cases where $U[t]$ has no elements, namely $t = 0$ and $t = \text{singleton}[0]$. Both of these are finite sets.

In[8]:= subclass[succ[singleton[0]], FINITE] // AssertTest
Out[8]= subclass[succ[singleton[0]], FINITE] = True

In[9]:= subclass[succ[singleton[0]], FINITE] := True

Note that:

In[10]:= equal[union[FINITE, succ[singleton[0]]], FINITE]
Out[10]= True

This justifies adding the following rewrite rule:

In[11]:= union[FINITE, succ[singleton[0]]] := FINITE

The main result of this section is the following restatement of these facts in which both of the variables have been eliminated.

In[12]:= Map[equal[V, class[t, equal[V, #]]] &, SubstTest[class, x, or[member[t, z], not[member[t, y]], not[member[x, u]], not[subclass[t, z]]], {u -> U[t], y -> PointClosed, z -> FINITE}]] // Reverse

In[13]:= subclass[intersection[PointClosed, P[FINITE]], FINITE] := True

Reintroducing the variable $t$ yields:

In[14]:= SubstTest[implies, and[member[t, u], subclass[u, v]], member[t, v], {u -> intersection[PointClosed, P[FINITE]], v -> FINITE}]
Out[14]= or[member[t, FINITE], not[member[t, PointClosed]], not[subclass[t, FINITE]]] = True

In[15]:= (% /. t_ -> t) /. Equal -> SetDelayed

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**some lemmas**

This section contains a collection of lemmas needed in the sequel.

Lemma 1: If $\text{image}[\text{RC}[U[t]], t]$ holds the empty set and all singletons contained in $U[t]$, and is closed under binary unions, then it holds every finite subset of $U[t]$.

In[16]:= SubstTest[implies, and[member[0, x], subclass[image[SINGLETON, y], x], subclass[image[CUP, cart[x, x]], x]], subclass[intersection[FINITE, P[y]], x], (x -> image[RC[U[t]], t], y -> U[t])]
Out[16]= or[not[member[U[t], t]], not[subclass[ image[CUP, cart[image[RC[U[t]], t], image[RC[U[t]], t]], image[RC[U[t]], t]], image[RC[U[t]], t]], subclass[intersection[FINITE, P[U[t]]], image[RC[U[t]], t]], image[RC[U[t]], t]]] = True
Lemma 2. If \( x \) is contained in \( y \), then their intersection is \( x \).

Lemma 3. If \( t \) is a set, and if every subset of \( U[t] \) is the relative complement of some member of \( t \), then \( t \) holds every subset of \( U[t] \).

Lemma 4. If \( t \) holds every subset of its sum class, then \( t \) is the power set of its sum class.

The proof carried out in the next section is based on the following facts:

Since a direct proof takes too long, the argument will be broken up into two steps and the hypotheses will be bundled so that it takes less time for the \texttt{GOEDEL} program to check the validity of the reasoning.
bundled and split

In the first part, the four of the hypotheses are bundled together. It is shown that these four hypotheses imply that every finite subset of \( U[t] \) is the relative complement of a member of \( t \).

\[
\text{t1-fin.nb}
\]

The rest of the argument is given here. What is shown is that a point-closed topology in which every open set is finite must be a discrete topology.

\[
\text{t1-fin.nb}
\]
\[ In[32] := \text{subclass}\{\text{intersection}[\text{T1}, \text{TOPS, P[FINE]}], \text{range[POWER]}\} := \text{True} \]