composite of a unary operation with itself

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If \( x \) is a unary operation, then so is \( \text{composite}[x, x] \). The derivation is a good illustration of the processes of eliminating and introducing variables.

**derivation**

The starting point is a result with variables:

\begin{verbatim}
In[2]:= SubstTest[implies, and[member[x, map[y, y]], member[z, map[y, y]]], member[composite[x, z], map[y, y]], z \rightarrow x]
Out[2]= or[member[composite[x, x], map[y, y]], not[member[x, map[y, y]]]] := True

In[3]:= or[member[composite[x, x], map[y, y]], not[member[x, map[y, y]]]] := True
\end{verbatim}

The variable \( x \) is eliminated first.

\begin{verbatim}
In[4]:= Map[equal[V, #] & , SubstTest[class, x, or[member[composite[x, x], z], not[member[x, z]]], z \rightarrow map[y, y]]] // Reverse
Out[4]= subclass[map[y, y], fix[image[inverse[COMPOSE], map[y, y]]]] = True

In[5]:= (% /. y \rightarrow y_) /. Equal \rightarrow SetDelayed
\end{verbatim}

One of the \( \text{map}[y,y] \) sets is easily replaced by the larger class \text{UNOPS}. \
\textbf{invariance under composite[COMPOSE, DUP]}

A related result is derived in this section. Again \texttt{reify} is used to remove the variable \texttt{y}.

\begin{verbatim}
In[12]:= Map[equal[0, #] \&, SubstTest[reify, y, 
        \{map[y, y], \texttt{fix}[image[\texttt{inverse}[COMPOSE], map[y, y]]], 
        \texttt{z} \rightarrow \texttt{COMPOSE}\}] \& \texttt{Reverse} 
\end{verbatim}

\texttt{Out[12]= subclass[\texttt{UNOPS}, \texttt{fix}[image[\texttt{inverse}[\texttt{COMPOSE}], \texttt{UNOPS}]]) = True}

\begin{verbatim}
In[13]:= subclass[\texttt{UNOPS}, \texttt{fix}[image[\texttt{inverse}[\texttt{COMPOSE}], \texttt{UNOPS}]]) := True
\end{verbatim}

The union of a collection of invariant subsets is itself invariant.

\begin{verbatim}
In[14]:= SubstTest[implies, subclass[u, \texttt{invar}[v]], \texttt{invar}[v, U[u]], 
        \{u \rightarrow \texttt{image}[	exttt{MAP, Id}]; \texttt{v} \rightarrow \texttt{composite}[\texttt{COMPOSE, DUP}]\}] 
\end{verbatim}

\texttt{Out[14]= subclass[\texttt{COMPOSE, id[U[UNOPS]]}, \texttt{UNOPS}] = True}

\begin{verbatim}
In[15]:= subclass[\texttt{COMPOSE, id[U[UNOPS]]}, \texttt{UNOPS}] := True
\end{verbatim}

This result is equivalent to the variable-free statement \texttt{subclass[UNOPS, \texttt{fix}[image[\texttt{inverse}[COMPOSE], \texttt{UNOPS}]])}

derived in the preceding section, but using direct images in place of inverse images.