open questions about range[IMAGE[x]]

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In[1]:= SetDirectory["l:"; << goedel.09aug25a

:Package Title: goedel.09aug25a 2009 August 25 at 12:40 noon

It is now: 2009 Aug 27 at 17:41

Loading Simplification Rules

In[2]:= SetDirectory["k://2009//aug//27"; << tools.m

TOOLS.M Revised 2009 August 27

weightlimit = 40

summary

When \(x\) is a set, the function IMAGE[x] is a proper class, and hence there is no function that takes \(x\) to IMAGE[x]. On the other hand, the range of the function IMAGE[x] is a set when \(x\) is a set, and hence there is a function that takes \(x\) to range[-IMAGE[x]]. Properties of this function are studied systematically in this notebook. Several open questions were encountered in the course of this work concerning whether various results that are true for sets extend to classes in general.

the function \(\lambda x.\) range[IMAGE[x]]

The function that takes \(x\) to range[IMAGE[x]] is:

In[3]:= VERTSECT[reify[x, range[IMAGE[x]]]]


The relation composite[IMG, inverse[FIRST]] that appears here has the following interpretation.

In[4]:= class[pair[u, v], exists[t, equal[v, image[u, t]]]]


The APPLY rule for the function VERTSECT[composite[IMG, inverse[FIRST]]] is as follows:

In[5]:= APPLY[VERTSECT[composite[IMG, inverse[FIRST]]], x]

Out[5]= union[complement[image[V, set[x]]], range[IMAGE[x]]]
**domain: a thinness result**

The statement that \( \text{range}\[\text{IMAGE}[x]\] \) is a set whenever \( x \) is a set can be reformulated without variables as a thin-ness condition for the relation \( \text{composite}[\text{IMG}, \text{inverse}[\text{FIRST}]] \).

**Theorem.** Domain of the function \( \text{VERTSECT}[\text{composite}[\text{IMG}, \text{inverse}[\text{FIRST}]]] \).

```plaintext
In[6]:=  Map[domain, SubstTest[reify, x, image, set[range[IMAGE[setpart[x]]]], v \rightarrow V]]
Out[6]=  domain[VERTSECT[composite[IMG, inverse[FIRST]]]] = V
```

**sum class theorem**

When \( x \) is a set, the sum class of \( \text{range}\[\text{IMAGE}[x]\] \) is the range of \( x \). More generally, one has:

```plaintext
In[8]:=  U[range[IMAGE[x]]]
Out[8]=  range[thinpart[x]]
```

A variable-free version of this fact can be derived.

**Theorem.**

```plaintext
In[9]:=  composite[BIGCUP, VERTSECT[composite[IMG, inverse[FIRST]]]] // ReifNormality
```

**image[inverse[S], range[IMAGE[x]]]**

**Theorem.**

```plaintext
In[11]:=  composite[IMAGE[inverse[S]], VERTSECT[composite[IMG, inverse[FIRST]]]] // ReifNormality:
           composite[POWER, IMAGE[SECOND]]
```

**Corollary.**
Counterexample. The `setpart` wrapper cannot be omitted.

```
In[15]:= equal[image[inverse[S], range[IMAGE[x_]]], P[range[x_]] /. x → E
Out[15]= False
```

Comment. It is not known at this point whether such an equation holds for all thin relations, or even for functions. It does hold when `x` is a bijection.

```
In[16]:= equal[image[inverse[S], range[IMAGE[x]]], P[range[x]] /. x → oopart[t]
Out[16]= True
```

---

**special cases**

For functions that are sets one has this result:

```
In[17]:= range[IMAGE[funpart[setpart[x]]]]
Out[17]= P[range[funpart[setpart[x]]]]
```

The variable-free reformulation of this result can be obtained using the new `FastReifNormality` test, which incorporates a `setpart` wrapper.

```
In[18]:= Begin["Goedel`Private`"];
In[19]:= ?? FastReifNormality
   a variant of ReifNormality with a setpart wrapper added
   FastReifNormality[x_] :=
   Module[{u = Unique[], v = Unique[]}, SubstTest[reify, v, image[u, set[setpart[v]]], u → x]]
```

Theorem.

```
In[20]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], FUNPART] // FastReifNormality
Out[20]= composite[POWER, IMAGE[SECOND], FUNPART]
```

```
In[21]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], FUNPART] :=
   composite[POWER, IMAGE[SECOND], FUNPART]
```

For bijections one need not assume `x` is a set.
In[22]:= range[IMAGE[oopart[x]]]
Out[22]= P[range[oopart[x]]]

Theorem. The result for bijections.

In[23]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], OOPART] // FastReifNormality
Out[23]= composite[VERTSECT[composite[IMG, inverse[FIRST]]], OOPART] :=
    composite[POWER, IMAGE[SECOND]], OOPART

In[24]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], OOPART] :=
    composite[POWER, IMAGE[SECOND]], OOPART

For the special case of the identity function \text{id}[x], one has:

In[25]:= range[IMAGE[id[x]]]
Out[25]= P[x]

A variable-free reformulation of this fact is the following.

Theorem.

In[26]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], IDP] // ReifNormality
Out[26]= composite[VERTSECT[composite[IMG, inverse[FIRST]]], IMAGE[DUP]] = POWER

In[27]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], IMAGE[DUP]] := POWER

For \text{HULL} functions, one has:

In[28]:= range[IMAGE[HULL[x]]]
Out[28]= P[fix[HULL[x]]]

Theorem.

In[29]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], LAMBHULL] // FastReifNormality
Out[29]= composite[VERTSECT[composite[IMG, inverse[FIRST]]], LAMBHULL] =
    composite[POWER, ACLOSURE]

In[30]:= composite[VERTSECT[composite[IMG, inverse[FIRST]]], LAMBHULL] :=
    composite[POWER, ACLOSURE]

The relative complement function \text{RC}[x] satisfies:

In[31]:= range[IMAGE[RC[setpart[x]]]]
Out[31]= P[P[setpart[x]]]

Theorem.
The class $\text{RS}[x]$ of restrictions of $x$ can be written as follows.

$$\text{range[IMAGE[composite[id[x], inverse[FIRST]]]]}$$

Observation.

$$\text{VERTSECT[reify[x, composite[id[x], inverse[FIRST]]]]}$$

Theorem.

$$\text{composite[VERTSECT[composite[IMG, inverse[FIRST]]]], IMAGE[composite[id[inverse[FIRST]], inverse[SECOND]]]}$$

The $\text{Uclosure}$ of $\text{range[IMAGE[x]]}$ is the same as that of $\text{range[VERTSECT[x]]}$.  Comment.  It is not known at present whether this result extends to classes in general.  No counterexample is immediately evident.

$$\text{Uclosure[range[IMAGE[setpart[x]]]]}$$

Theorem.

$$\text{composite[UCLOSURE, VERTSECT[composite[IMG, inverse[FIRST]]]]}$$

The $\text{Uclosure}$ of $\text{range[IMAGE[x]]}$ is the same as that of $\text{range[VERTSECT[x]]}$.  Comment.  It is not known at present whether this result extends to classes in general.  No counterexample is immediately evident.
\textbf{In}[41]:= \text{Uclosure[range[IMAGE[x]]]}

\textbf{Out}[41]= \text{Uclosure[range[VERTSECT[x]]]}

For the case of sets, this simplifies:

\textbf{In}[42]:= \text{Uclosure[range[VERTSECT[setpart[x]]]]}

\textbf{Out}[42]= \text{range[IMAGE[setpart[x]]]}

Theorem.

\textbf{In}[43]:= \text{composite[UCLOSURE, IMAGE[SECOND], VS] // FastReifNormality}

\textbf{Out}[43]= \text{composite[UCLOSURE, IMAGE[SECOND], VS] = VERTSECT[composite[IMG, inverse[FIRST]]]}

\textbf{In}[44]:= \text{composite[UCLOSURE, IMAGE[SECOND], VS] := VERTSECT[composite[IMG, inverse[FIRST]]]}

ra-image.nb