summary

The APPLY rule for rotate[x] is removed now and will be replaced with two rewrite rules in the opposite direction.

As an application, six basic equations in the theory of quasigroups are derived.

replacement rules

Theorem.

In [3] := SubstTest[A, image[t, set[PAIR[y, z]]], t \rightarrow rotate[inverse[x]]] // Reverse


In [4] := APPLY[image[x_, set[y_], z_] := APPLY[rotate[inverse[x]], PAIR[y, z]]

Theorem.

In [5] := SubstTest[APPLY, image[t, set[y]], z, t \rightarrow composite[SWAP, x]] // Reverse

Out[5] = APPLY[inverse[image[x, set[y]]], z] = APPLY[rotate[composite[inverse[x], SWAP]], PAIR[y, z]]

In [6] := APPLY[inverse[image[x_, set[y_]]], z_] := APPLY[rotate[composite[inverse[x], SWAP]], PAIR[y, z]]
NATADD rule

Theorem.

\[\text{In[7]} := \text{SubstTest[APPLY, image[inverse[t], set[x]], y, t \rightarrow \text{NATADD}]}\]

\[\text{Out[7]} = \text{APPLY[rotate[NATADD], PAIR[x, y]] = natsub[x, y]}\]

\[\text{In[8]} := \text{APPLY[rotate[NATADD], PAIR[x_, y_] := natsub[x, y]}\]

quasigp equations

If elements \(a, b, c\) of a quasigroup satisfy \(c = a \cdot b = c\), then any two of these determines the third. One writes: \(a = c / b\) and \(b = a \setminus c\). The following abbreviations help to compare with the formulas in books on quasigroups.

\[\text{In[9]} := \text{into[x_]} := \text{flip[rotate[x]}\]

\[\text{In[10]} := \text{over[x_]} := \text{rotate[flip[x]}\]

If \(q\) is a quasigroup binary operation, then \(q, \text{into}[q]\) and \(\text{over}[q]\) are also quasigroup binary operations. The notation \(a \cdot b = c\) is equivalent to the statement \(\text{member[pair[pair[a, b], c], q]}\). Similarly, the notations \(a \setminus c = b\) and \(c / b = a\) are equivalent to the statements:

\[\text{In[11]} := \text{member[pair[pair[a, c], b], \text{into}[q]}\]

\[\text{Out[11]} = \text{member[pair[pair[a, b], c], q]}\]

\[\text{In[12]} := \text{member[pair[pair[c, b], a], \text{over}[q]}\]

\[\text{Out[12]} = \text{member[pair[pair[a, b], c], q]}\]

One can use each of these three equations to eliminate one of the three variables in the other two equations. This yields a total of six equations:

- (IL) \(a \setminus (a \cdot b) = b\)
- (IR) \(a = (a \cdot b) / b\)
- (SL) \(a \cdot (a \setminus c) = c\)
- (SR) \(c = (c / b) \cdot b\)
- (DL) \(c / (a \setminus c) = a\)
- (DR) \(b = (c / b) \setminus c\)

See, for example, page 3 in the following reference for the first four of these six equations. The remaining two are on page 6.
The letters used in the abbreviations used for these equations are shorthand for I = injective, S = surjective, D = division and L = left.

R = right. Each one of these six equations corresponds to a rewrite rule in the GOEDEL program for the wrapper \texttt{quasigp[x]}. Only the first equation needs to be derived; the other five can be obtained using \texttt{flip} and \texttt{rotate}.

Theorem. Equation (IL). This equation is derived using \texttt{ReifNormality}.

\begin{verbatim}
\end{verbatim}

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Theorem. Equation (IL). This equation is derived using \texttt{ReifNormality}.

\begin{verbatim}
In[14]:= Map[\texttt{APPLY}, \texttt{PAIR[y, z]}] \\
     & \texttt{composite[rotate[quasigp[x]]]}, \\
     \texttt{cross[composite[quasigp[x]], SWAP], Id], ROT, cross[DUP, Id]} /\texttt{ReifNormality}\n\end{verbatim}

Corollary. Equation (IR).

\begin{verbatim}
In[15]:= \texttt{APPLY[rotate[quasigp[x_]], PAIR[\texttt{PAIR[y, z]}], \texttt{PAIR[y_, z_], y_]]} := \\
     \texttt{union[z, complement[image[V, intersection\texttt{range[quasigp[x]]}, \texttt{set[y]}]]],} \\
     \texttt{complement[image[V, intersection\texttt{range[quasigp[x]]}, \texttt{set[z]}]]]}\n\end{verbatim}

Corollary. Equation (SL).

\begin{verbatim}
In[16]:= \texttt{SubstTest[\texttt{APPLY}, \texttt{rotate[quasigp[t]]},} \\
     \texttt{PAIR[\texttt{PAIR[quasigp[t, PAIR[y, z]], y] \rightarrow flip[quasigp[x]]}] /\texttt{Reverse}\n\end{verbatim}

Corollary. Equation (SR).

\begin{verbatim}
In[17]:= \texttt{APPLY[\texttt{rotate[composite[quasigp[x_], SWAP]]},} \\
     \texttt{PAIR[\texttt{PAIR[quasigp[x_], PAIR[z_, y_]], y_]]} := \\
     \texttt{union[z, complement[image[V, intersection\texttt{range[quasigp[x]]}, \texttt{set[y]}]]],} \\
     \texttt{complement[image[V, intersection\texttt{range[quasigp[x]]}, \texttt{set[z]}]]]}\n\end{verbatim}
Corollary. Equation (DL).

\[ \text{In[22]} := \text{SubstTest[APPLY, rotate[quasigp[t]],}
\text{PAIR[APPLY[quasigp[t], PAIR[y, z]], y], t \to rotate[quasigp[x]]]} \] // Reverse

\[ \text{Out[22]} = \text{APPLY[quasigp[x]], PAIR[APPLY[rotate[composite[quasigp[x], SWAP]], PAIR[z, y]], y]} =
\text{union[z, complement[image[V, intersection[range[quasigp[x]], set[y]]]]],}
\text{complement[image[V, intersection[range[quasigp[x]], set[z]]]]} \]

\[ \text{In[23]} := \text{APPLY[rotate[composite[quasigp[x], SWAP]],}
\text{PAIR[y, APPLY[rotate[quasigp[x]], PAIR[y, z]]]] =}
\text{union[z, complement[image[V, intersection[range[quasigp[x]], set[y]]]]],}
\text{complement[image[V, intersection[range[quasigp[x]], set[z]]]]} \]

Corollary. Equation (DR).

\[ \text{In[24]} := \text{SubstTest[APPLY, rotate[quasigp[t]],}
\text{PAIR[APPLY[quasigp[t], PAIR[y, z]], y], t \to rotate[flip[quasigp[x]]]} \] // Reverse

\[ \text{Out[24]} = \text{APPLY[rotate[quasigp[x]],}
\text{PAIR[y, APPLY[rotate[composite[quasigp[x], SWAP]], PAIR[y, z]]]] =}
\text{union[z, complement[image[V, intersection[range[quasigp[x]], set[y]]]]],}
\text{complement[image[V, intersection[range[quasigp[x]], set[z]]]]} \]

\[ \text{In[25]} := \text{APPLY[rotate[composite[quasigp[x], SWAP]], PAIR[y, z]]} :=
\text{union[z, complement[image[V, intersection[range[quasigp[x]], set[y]]]]],}
\text{complement[image[V, intersection[range[quasigp[x]], set[z]]]]} \]