the class of associative relations is a proper class

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In[1]:= << goedel52.s31; << tools.m

:Package Title: goedel52.s31 2003 July 2 at 9:50 a.m.
It is now: 2003 Jul 8 at 9:22
Loading Simplification Rules
TOOLS.M Revised 2003 July 6
weightlimit = 40

■ summary

It is shown in this notebook that the class ASSOCIATIVE of all associative relations is a proper class.

■ derivation

Most of the work is in the first step. This amounts to the observation that cart[cart[x,x],y] is always associative.

In[2]:= image[inverse[cross[composite[CART, DUP], Id]],
image[inverse[CART], ASSOCIATIVE]] // RelnNormality

Out[2]= composite[image[inverse[CART], ASSOCIATIVE], CART, DUP] == cart[V, V]

The result can be reformulated in the following more familiar fashion:

In[3]:= Map[equal[0, composite[Id, complement[#]]] &, %]


In[4]:= subclass[image[CART, cart[image[CART, Id], V]], ASSOCIATIVE] := True

■ the sum class of ASSOCIATIVE

Lemma:

In[5]:= SubstTest[implies, subclass[u, v], subclass[U[u], U[v]],
{u -> image[CART, cart[image[CART, Id], V]], v -> ASSOCIATIVE}]

Out[5]= subclass[cart[cart[V, V], V], U[ASSOCIATIVE]] == True

In[6]:= subclass[cart[cart[V, V], V], U[ASSOCIATIVE]] := True
The opposite inclusion also holds:

\[
\text{In}[7]:= \text{Map}[\text{equal}[0, \#] \&, \text{dif}[\text{ASSOCIATIVE}, \text{P}[\text{cart}[\text{cart}[V, V], V]]] // \text{Normality}
\]

\[
\text{Out}[7]= \text{subclass}[\text{U}[\text{ASSOCIATIVE}], \text{cart}[\text{cart}[V, V], V]] == \text{True}
\]

\[
\text{In}[8]:= \text{subclass}[\text{U}[\text{ASSOCIATIVE}], \text{cart}[\text{cart}[V, V], V]] := \text{True}
\]

The two inclusions can be combined into an equation:

\[
\text{In}[9]:= \text{SubstTest}[\text{and}, \text{subclass}[u, v], \text{subclass}[v, u],
\{u -> \text{cart}[\text{cart}[V, V], V], v -> \text{U}[\text{ASSOCIATIVE}]\}]
\]

\[
\text{Out}[9]= \text{True} == \text{equal}[\text{cart}[\text{cart}[V, V], V], \text{U}[\text{ASSOCIATIVE}]]
\]

The following rewrite rule will be permanently retained:

\[
\text{In}[10]:= \text{U}[\text{ASSOCIATIVE}] := \text{cart}[\text{cart}[V, V], V]
\]

**ASSOCIATIVE is a proper class**

As a corollary, one obtains the fact that ASSOCIATIVE is a proper class.

\[
\text{In}[11]:= \text{SubstTest}[\text{member}, \text{U}[x], V, x -> \text{ASSOCIATIVE}] // \text{Reverse}
\]

\[
\text{Out}[11]= \text{member}[\text{ASSOCIATIVE}, V] == \text{False}
\]

\[
\text{In}[12]:= \text{member}[\text{ASSOCIATIVE}, V] := \text{False}
\]

The following is a better rewrite rule:

\[
\text{In}[13]:= \text{member}[\text{ASSOCIATIVE}, x] // \text{AssertTest}
\]

\[
\text{Out}[13]= \text{member}[\text{ASSOCIATIVE}, x] == \text{False}
\]

\[
\text{In}[14]:= \text{member}[\text{ASSOCIATIVE}, x_] := \text{False}
\]