inclusions for \texttt{binclosed}[x] and \texttt{invar}[x]

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\texttt{In[1]:= SetDirectory["l:"}; \texttt{<< goedel.13oct12a}

:Package Title: goedel.13oct12a 2013 October 12 at 7:20 p.m.
Loading takes about seventeen minutes, half that time due to builtin pauses.

It is now: 2013 Oct 13 at 12:40
Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3
weightlimit = 40
Loading completed.

It is now: 2013 Oct 13 at 12:56

\textbf{summary}

Rewrite rules for inclusions involving \texttt{binclosed}[x] and \texttt{invar}[x] are derived.

\textbf{introduction}

If a class is invariant under both \texttt{x} and \texttt{y}, then it is also invariant under their composite \texttt{x \circ y}. The following is an example.

Theorem.

\texttt{In[5]:= SubstTest[subclass, intersection[invar[x], invar[y]],}
\texttt{ invar[composite[x, y]], \{x \to IMAGE[\texttt{FIRST}], y \to IMAGE[\texttt{SWAP}])]} // Reverse

\texttt{ invar[IMAGE[\texttt{SECOND}]]]} = \texttt{True}

\texttt{In[6]:= subclass[intersection[invar[IMAGE[\texttt{FIRST}]], invar[IMAGE[\texttt{SWAP}]]],}
\texttt{ invar[IMAGE[\texttt{SECOND}]}} = \texttt{True}

Similar results can be derived for classes of sets that are binary closed under given relations. In this notebook some basic rules are derived that can be used to show that sets that are binary closed or invariant under given relation are also binary closed under other relations that can be constructed from them.
invar[SINGLETON]

Binary closure under \( x \) implies invariance under \( x \circ \text{DUP} \). The following is an example.

Theorem. Binary closure under \( \text{PAIRSET} \) implies invariance under \( \text{SINGLETON} \).

\[
\begin{align*}
\text{In}[7]:=& \quad \text{SubstTest}\@\text{subclass, binclosed}[\text{x}, \text{invar}[\text{composite}[\text{x}, \text{DUP}]], \text{x}\rightarrow\text{PAIRSET}] \quad \text{// Reverse} \\
\text{Out}[7]=& \quad \text{subclass}[\text{binclosed}[\text{PAIRSET}], \text{invar}[\text{SINGLETON}]] = \text{True} \\
\text{In}[8]:=& \quad \text{subclass}[\text{binclosed}[\text{PAIRSET}], \text{invar}[\text{SINGLETON}]] := \text{True}
\end{align*}
\]

binclosed[CUP]

If a class is invariant under \( x \) and binary closed under \( y \), then it is binary closed under \( x \circ y \).

Theorem. If a set is invariant under \( \text{BIGCUP} \) and binary closed under \( \text{PAIRSET} \), then it is binary closed under \( \text{CUP} \).

\[
\begin{align*}
\text{In}[9]:=& \quad \text{SubstTest}\@\text{subclass, intersection}[\text{invar}[\text{x}], \text{binclosed}[\text{y}]], \\
& \quad \text{binclosed}[\text{composite}[\text{x}, \text{y}]], \{\text{x}\rightarrow\text{BIGCUP}, \text{y}\rightarrow\text{PAIRSET}\} \quad \text{// Reverse} \\
\text{Out}[9]=& \quad \text{subclass}[\text{intersection}[\text{binclosed}[\text{PAIRSET}], \text{invar}[\text{BIGCUP}]], \text{binclosed}[\text{CUP}]] = \text{True} \\
\text{In}[10]:=& \quad \text{subclass}[\text{intersection}[\text{binclosed}[\text{PAIRSET}], \text{invar}[\text{BIGCUP}]], \text{binclosed}[\text{CUP}]] := \text{True}
\end{align*}
\]

binclosed is antitone

If \( x \subset y \), then \( \text{binclosed}[y] \subset \text{binclosed}[x] \).

Theorem.

\[
\begin{align*}
\text{In}[11]:=& \quad \text{SubstTest}\@\text{implies, subclass}[\text{u}, \text{v}], \\
& \quad \text{subclass}[\text{binclosed}[\text{v}], \text{binclosed}[\text{u}]], \{\text{u}\rightarrow\text{intersection}[\text{x}, \text{y}], \text{v}\rightarrow\text{x}\} \quad \text{// Reverse} \\
\text{Out}[11]=& \quad \text{subclass}[\text{binclosed}[\text{x}], \text{binclosed}[\text{intersection}[\text{x}, \text{y}]]] = \text{True} \\
\text{In}[12]:=& \quad \text{subclass}[\text{binclosed}[\text{x}]], \text{binclosed}[\text{intersection}[\text{x}, \text{y}]]] := \text{True}
\end{align*}
\]

Theorem.

\[
\begin{align*}
\text{In}[13]:=& \quad \text{SubstTest}\@\text{implies, subclass}[\text{u}, \text{v}], \\
& \quad \text{subclass}[\text{binclosed}[\text{v}], \text{binclosed}[\text{u}]], \{\text{u}\rightarrow\text{composite}[\text{x}, \text{id}[\text{y}]], \text{v}\rightarrow\text{x}\} \quad \text{// Reverse} \\
\text{Out}[13]=& \quad \text{subclass}[\text{binclosed}[\text{x}], \text{binclosed}[\text{composite}[\text{x}, \text{id}[\text{y}]]]] = \text{True} \\
\text{In}[14]:=& \quad \text{subclass}[\text{binclosed}[\text{x}], \text{binclosed}[\text{composite}[\text{x}, \text{id}[\text{y}]]]] := \text{True}
\end{align*}
\]
Theorem. If a set is binary closed under $x$ and invariant under $y$ and $z$, then it is binary closed under $x \circ (y \otimes z)$.

Theorem. If a set is binary closed under $x$ and invariant under $y$ and $z$, then it is binary closed under $x \circ (y \otimes z)$.

Lemma.
Theorem. If a set is binary closed under $x$, $y$ and $z$, then it is also binary closed under $x \circ (y \otimes z) \circ \text{DUP}$.

Theorem. (An application to \textsc{Symdiff}.)