category unit laws

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In[1]:= SetDirectory["l:"; \<< goedel.09jan05a; \<< tools.m

:Package Title: goedel.09jan05a 2009 January 5 at 4:00 p.m.

It is now: 2009 Jan 6 at 15:35

Loading Simplification Rules

TOOLS.M Revised 2008 December 26

weightlimit = 40

summary

For MacLane, an abstract objects-and-arrows category consists of five classes, subject to various axioms. MacLane’s five classes are a class of morphisms, a subclass of the class of morphisms called the class of identity morphisms (identified with objects), an associative function called the product (or composition) law, and two functions called domain and codomain. In the GOEDEL program only the product law is considered basic, and the other four classes are defined in terms of it. We use the cat[x] wrapper to denote the product law. The class of morphisms is defined to be range[cat[x]], the class of identity morphisms is ids[cat[x]], and the domain and codomain functions are dom[cat[x]] and cod[cat[x]], respectively.

In[2]:="Saunders MacLane, Categories for the Working Mathematician, Springer-Verlag, New York, 1971."

In the comments only, the following abbreviations will be used: mor = range[cat[x]], ids = ids[cat[x]], prod = cat[x], dom = dom[cat[x]] and cod = cod[cat[x]]. MacLane’s notation f: a \rightarrow b means:

\[ \text{and[member[f, mor], member[pair[f, a], dom], member[pair[f, b], cod]]} \]

Among MacLane’s objects-and-arrows axioms for category theory are what he calls the identity and unit laws. In this notebook rewrite rules are derived that correspond to these laws.

MacLane’s identity law

MacLane takes as an axiom the Identity Law: Every identity is a morphism with the same domain and codomain (see MacLane, page 7): \( 1_a: a \rightarrow a \). This assumption is here translated into the two statements \( \text{fix[dom]} = \text{ids} \) and \( \text{fix[cod]} = \text{ids} \). These equations are derived here from the definitions of dom and cod.
Theorem. The fixed-point class of the \texttt{dom} function is equal to the class of identity morphisms.

\begin{verbatim}
In[3]:= SubstTest[fix, composite[id[u], v], \{u \rightarrow ids[cat[x]], v \rightarrow domain[cat[x]]\}] // Reverse
Out[3]= fix[dom[cat[x]]] = ids[cat[x]]
\end{verbatim}

Corollary. The fixed-point class of the \texttt{cod} function is equal to the class of identity morphisms.

\begin{verbatim}
In[5]:= SubstTest[fix, dom[cat[t]], t \rightarrow flip[cat[x]]] // Reverse
Out[5]= fix[cod[cat[x]]] = ids[cat[x]]
\end{verbatim}

\section*{MacLane's unit laws}

Lemma.

\begin{verbatim}
In[7]:= SubstTest[empty, composite[Id, dif[u, v]],
    \{u \rightarrow composite[\texttt{FIRST}, id[domain[u]]], v \rightarrow composite[\texttt{cat[x]}, id[domain[\texttt{cat[x]]]]]\}]
Out[7]= subclass[composite[\texttt{FIRST}, id[domain[u]]], \texttt{cat[x]}] = True
\end{verbatim}

Theorem. Equational form of the second unit law.

\begin{verbatim}
In[9]:= SubstTest[implies, and subclass[u, v], \texttt{FUNCTION[v]}],
    equal[u, composite[v, id[domain[u]]]],
    \{u \rightarrow composite[\texttt{FIRST}, id[domain[\texttt{cat[x]}]]], v \rightarrow \texttt{cat[x]}\}] // Reverse
Out[9]= equal[composite[\texttt{FIRST}, id[domain[\texttt{cat[x]}]]], composite[\texttt{cat[x]}, id[domain[\texttt{cat[x]}]]]] = True
\end{verbatim}

Corollary. Equational form of the first unit law.

\begin{verbatim}
In[11]:= Map[\texttt{flip}, SubstTest[composite, cat[t], id[domain[\texttt{cat[t]}]]], t \rightarrow \texttt{flip[cat[x]]}] // Reverse
    composite[\texttt{SECOND}, id[inverse[cod[\texttt{cat[x]}]]]]
\end{verbatim}

MacLane takes as an axiom the Second Unit law (see MacLane, page 8): if \texttt{g: b \rightarrow c}, then \texttt{g \cdot 1_b = g}.

Theorem. The composite of a morphism \texttt{u} with its \texttt{dom}-identity morphism is \texttt{u}. 

\begin{verbatim}
In[12]:= composite[\texttt{cat[x]}, id[inverse[cod[\texttt{cat[x]}]]]] :=
    composite[\texttt{SECOND}, id[inverse[cod[\texttt{cat[x]}]]]]
\end{verbatim}
MacLane takes as an axiom the First Unit law (see MacLane, page 8): if \( f: a \rightarrow b \), then \( 1_b \cdot f = f \).

Corollary. A similar result for the cod-identity morphism obtained by using duality.