complete lattice restrictions of S

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2009 February 26

It is shown that if a set \( x \) has a greatest member and is closed under arbitrary intersections, then the set \( x \), partially ordered by inclusion, is a complete lattice order. This result is dual to a similar one for unions that is already available:

\[
\text{In[2]} := \text{implies}[	ext{member}[x, \text{fix}[\text{UCLOSURE}]], \text{member}[	ext{composite}[\text{id}[x], S, \text{id}[x]], \text{CL}]]
\]

\[
\text{Out[2]} = \text{True}
\]

**derivation**

Lemma. Consequence of a criterion for being a complete lattice.

\[
\text{In[3]} := \text{SubstTest}[\text{and}, \text{member}[t, \text{PO}], \text{subclass}[P[\text{fix}[t]], \text{domain}[\text{GLB}[t]]], t \rightarrow \text{restrict}[S, x, x]] // \text{Reverse}
\]

\[
\text{Out[3]} = \text{and}[\text{member}[x, V], \text{subclass}[P[x], \text{domain}[\text{GLB}[	ext{composite}[\text{id}[x], S, \text{id}[x]]]]], \text{member}[	ext{composite}[\text{id}[x], S, \text{id}[x]], \text{CL}]
\]

\[
\text{In[4]} := \text{and}[\text{member}[x_\_], V], \text{subclass}[P[x_\_], \text{domain}[\text{GLB}[	ext{composite}[\text{id}[x_\_], S, \text{id}[x_\_]]]]], \text{member}[	ext{composite}[\text{id}[x], S, \text{id}[x]], \text{CL}]
\]

Lemma. A condition for \( \text{restrict}[S, x, x] \) to be a complete lattice.
Theorem. If \( x \) has a greatest member, and is closed under arbitrary intersections, then \( \text{restrict} [S, x, x] \) is a complete lattice order.

Lemma. A simplification rule.

eliminating variables

Lemma. Eliminating the variable \( x \).
Theorem. Variable-free restatement of the fact that restrictions of S to intersection-closed sets having greatest members are complete lattices.

In[16]:= SubstTest[implies, subclass[u, v],
    subclass[image[t, u], image[t, v]],
    \{t \to \text{composite}[\text{IMAGE}[\text{id}[S]], \text{CART, DUP}],
    u \to \text{intersection}[\text{fix}[\text{ACLOSURE}], \text{fix}[\text{composite}[\text{E}, \text{BIGCUP}]]],
    v \to \text{fix}[\text{image}[\text{inverse}[\text{CART}], \text{image}[\text{inverse}[\text{IMAGE}[\text{id}[S]], \text{CL}]]])] // Reverse

Out[16]= subclass[image[\text{IMAGE}[\text{id}[S]]], \text{image}[\text{CART, id[intersection[\text{fix}[\text{ACLOSURE}], \text{fix}[\text{composite}[\text{E}, \text{BIGCUP}]]]]]]], \text{CL} \Leftarrow \text{True}

Theorem. Restrictions of the subset relation S to sets closed under arbitrary unions are complete lattice orders.

In[19]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
    \{t \to \text{composite}[\text{IMAGE}[\text{id}[S]], \text{CART, DUP}],
    u \to \text{fix}[\text{UCLOSURE}],
    v \to \text{fix}[\text{image}[\text{inverse}[\text{CART}], \text{image}[\text{inverse}[\text{IMAGE}[\text{id}[S]], \text{CL}]]])] // Reverse

Out[19]= subclass[image[\text{IMAGE}[\text{id}[S]]], \text{image}[\text{CART, id[fix[\text{UCLOSURE}]]}], \text{CL} \Leftarrow \text{True}