topologies characterized via closed sets

Johan G. F. Belinfante
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In[1]:= SetDirectory["l:"]; << goedel.11feb04a

:Package Title: goedel.11feb04a 2011 February 4 at 1:50 p.m.

It is now: 2011 Feb 7 at 13:37

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

summary

A topology can be characterized by its class of closed sets. This is the topic of Theorem 4 on page 40 of the following reference. (Kelley dismisses it as too trivial to be given a formal proof.)


Kelley’s Theorem 4 says that if \( x \) satisfies the four conditions \( \text{Aclosure}[x] = x \), \( \text{image}[\text{CUP}, \text{cart}[x, x]] = x \), \( U[x] \in x \) and \( 0 \in x \), then \( x \) is the class of closed sets for the topology \( t = \text{image}[\text{RC}[U[x]], x] \).

the function \( \text{IRC} = \lambda x. \text{image}[\text{RC}[U[x]], x] \)

Variable-free statements about classes of closed sets can be derived by introducing the function \( \lambda x. \text{image}[\text{RC}[U[x]], x] \) for which the acronym \( \text{IRC} \) is introduced, standing for "image under relative complements."

In[5]:= composite[IMG, id[composite[inverse[BIGCUP], inverse[RCF]]], inverse[SECOND]] := IRC

Theorem. \( \text{IRC} \) is a function.

In[6]:= SubstTest[FUNCTION, composite[funpart[t],
    id[composite[inverse[BIGCUP], inverse[RCF]]], inverse[SECOND]], t -> IMG] // Reverse


In[7]:= FUNCTION[IRC] := True

Theorem. The domain of \( \text{IRC} \) is the class of all sets.
Theorem. An APPLY rule for IRC.

```
In[10]:=  Apply[IMG, composite[
            id[composite[inverse[BIGCUP], inverse[RCF]]], inverse[SECOND]], x] // Reverse
```

```
```

Theorem. A vertical section rule.

```
In[12]:=  ImageComp[IMG, 
            composite[id[composite[inverse[BIGCUP], inverse[RCF]]], inverse[SECOND]], set[x]]
```

```
Out[12]=  image[IRC, set[x]] := intersection[image[V, set[x]], set[image[RC[U[x]], x]]]
```

Theorem. Inverse image rule.

```
In[14]:=  (member[x, image[inverse[funpart[t]], y]] // AssertTest) /. t -> IRC
```

```
Out[14]=  member[x, image[inverse[IRC], y]] := and[member[x, V], member[image[RC[U[x]], x], y]]
```

```
In[15]:=  member[x_, image[inverse[IRC], y_]] := and[member[x, V], member[image[RC[U[x]], x], y]]
```

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**introduction**

A topology is a set of sets which is closed under binary intersections and arbitrary unions. The class of all topologies is called TOPS. By definition, the collection of closed sets of a topology t is image[RC[U[t]], t]. The topology itself can be recovered from the class of closed sets by the same operation. This result is currently recognized by the GOEDEL program when the top[t] wrapper is used for a topology. One can remove the wrapper as follows.

```
In[3]:=  SubstTest[implies, and[equal[x, top[t]], equal[y, image[RC[U[x]], x]]], 
          equal[x, image[RC[U[y]], y]], t -> x] // Reverse
```

```
Out[3]=  or[equal[x, image[RC[U[y]], y]], 
          not[equal[y, image[RC[U[x]], x]]], not[member[x, TOPS]]] = True
```

```
In[4]:=  (% /. {x -> x_, y -> y_}) /. Equal -> SetDelayed
```

A variable-free formulation of this result can be derived as follows.
Theorem. \(\text{id[TOPS] \circ \text{inverse[IRC]} \subseteq \text{IRC}}\).

\begin{verbatim}
In[16]:= Map[empty[range[composite[Id, complement[#]]]] &, SubstTest[class, pair[x, y], implies[and[member[setpart[x], t], equal[setpart[y], APPLY[z, setpart[x]]]]], equal[setpart[x], APPLY[z, setpart[y]]], {t \rightarrow TOPS, z \rightarrow IRC}]]

Out[16]= subclass[composite[id[TOPS], inverse[IRC]], IRC] = True
\end{verbatim}

In[17]:= \%/\$. Equal \rightarrow \text{SetDelayed}

Corollary. \(\text{id[TOPS] \circ \text{inverse[IRC]} \text{ is a function.}}\)

\begin{verbatim}
In[18]:= SubstTest[implies, and[subclass[u, v], FUNCTION[v]],
FUNCTION[u], {u \rightarrow composite[id[TOPS], inverse[IRC]], v \rightarrow IRC}] // Reverse

Out[18]= FUNCTION[composite[id[TOPS], inverse[IRC]]] = True

In[19]:= FUNCTION[composite[id[TOPS], inverse[IRC]]] := True
\end{verbatim}

a preliminary result

One of the hypotheses in Kelley’s Theorem 4 is the class \(x\) of closed sets of a topology is closed under finite unions. This condition is rewritten by the GOEDEL program as follows.

\begin{verbatim}
In[20]:= equal[image[BIGCUP, intersection[FINITE, P[x]]], x]

Out[20]= and[member[0, x], subclass[image[CUP, cart[x, x]], x]]
\end{verbatim}

In this section only the first part of Kelley’s Theorem 4 is considered. This part says that if \(x\) satisfies the four conditions \(\text{Aclosure}[x] = x\), \(\text{image}[CUP, \text{cart}[x, x]] = x\), \(U[x] \in x\) and \(0 \in x\), then \(t = \text{image}[\text{RC}[U[x]], x]\) is a topology. For this part, the condition \(0 \in x\) is not needed. Three lemmas are needed.

The first lemma deals with the issue that \(\text{Aclosure}[x]\) and \(\text{fix[HULL}[x]]\) only agree when \(x\) is a set.

Lemma.

\begin{verbatim}
In[21]:= Map[not, SubstTest[and, implies[and[p2, p3], p4],
not[implies[and[p1, p2], p4]], {p1 \rightarrow member[U[x], x], p2 \rightarrow equal[x, \text{Aclosure}[x]],
p3 \rightarrow member[x, V], p4 \rightarrow equal[x, \text{fix[HULL}[x]]]}]] // Reverse

Out[21]= \%[equal[x, \text{fix[HULL}[x]]], not[equal[x, \text{Aclosure}[x]]], not[member[U[x], x]]] = True

In[22]:= \%[equal[x_, \text{fix[HULL}[x_]]],
not[equal[x_, \text{Aclosure}[x_]]], not[member[U[x_], x_]]] := True
\end{verbatim}

The second lemma just applies the first lemma to the case at hand.

Lemma.
The third lemma rewrites relative complements of intersections of relative complements as unions.

Lemma.

Theorem. If $x$ is a collection of sets that is binary closed under unions, closed under arbitrary intersections, and holds its sum class, then the set of relative complements of members of $x$ in $U[x]$ is a topology.

A variable-free formulation of this result can be derived.

Theorem.
**a counter-example**

The three conditions \( \text{Aclosure}[x] = x \), \( \text{image}[\text{CUP}, \text{cart}[x, x]] = x \) and \( U[x] \in x \) do not suffice to characterize the class of closed sets for a topology. For any topology, the class of all closed sets holds the empty set.

\[
\text{In}[31]:= \text{implies}[\text{member}[t, \text{TOPS}], \text{member}[0, \text{image}[\text{RC}[U[t]], t]]]
\]

\[
\text{Out}[31]= \text{True}
\]

The three conditions on \( x \) in the theorem in the preceding section do not imply \( 0 \in x \).

Counterexample. The three conditions on \( x \) in the theorem in the preceding section do not imply \( 0 \in x \).

\[
\text{In}[32]:= \text{implies}[\text{and}[\text{equal}[\text{Aclosure}[x], x], \text{equal}[\text{image}[\text{CUP}, \text{cart}[x, x]], x], \text{member}[U[x], x]], \text{member}[0, x]]/. x \rightarrow \text{set}[\text{set}[0]]
\]

\[
\text{Out}[32]= \text{False}
\]

Kelley’s Theorem 4 says that if \( x \) satisfies all four conditions \( \text{Aclosure}[x] = x \), \( \text{image}[\text{CUP}, \text{cart}[x, x]] = x \), \( U[x] \in x \) and \( 0 \in x \), then \( x \) is the class of closed sets for the topology \( t = \text{image}[\text{RC}[U[x]], x] \). Eliminating the variable \( t \) from the conclusion yields the following messy statement.

\[
\text{In}[33]:= \text{equal}[x, \text{image}[\text{RC}[U[t]], t]]/. t \rightarrow \text{image}[\text{RC}[U[x]], x]
\]

\[
\text{Out}[33]= \text{equal}[x, \text{image}[\text{RC}[\text{intersection}[\text{complement}[A[x]], \text{image}[V, \text{set}[x]], U[x]], \text{image}[\text{RC}[U[x]], x]]]]
\]

Observe that since \( x \) is a set, \( \text{image}[V, \text{set}[x]] = V \).

\[
\text{In}[34]:= \text{implies}[\text{member}[U[x], x], \text{equal}[\text{image}[V, \text{set}[x]], V]]
\]

\[
\text{Out}[34]= \text{True}
\]

Observe further that since \( 0 \in x \), one has \( A[x] = 0 \).

\[
\text{In}[35]:= \text{implies}[\text{member}[0, x], \text{equal}[0, A[x]]]
\]

\[
\text{Out}[35]= \text{True}
\]

With these simplifications, the conclusion reduces to the statement that \( x \) is a set.

\[
\text{In}[36]:= (\text{equal}[x, \text{image}[\text{RC}[U[t]], t]]/. t \rightarrow \text{image}[\text{RC}[U[x]], x])/. \{\text{image}[V, \text{set}[x]] \rightarrow V, A[x] \rightarrow 0\}
\]

\[
\text{Out}[36]= \text{member}[x, V]
\]

Although this does informally prove Kelley’s Theorem 4, it is not provide a convenient formal statement of the theorem. This defect can be overcome by using the function \( \text{IRC} \).
**image[IRC, TOPS]**

A set \( x \) belongs to the class \( \text{image[IRC, TOPS]} \) if it is the class of closed sets for some topology. One can use \texttt{reify} to quickly derive variable-free formulations of standard results about closed sets.

**Theorem.** The collection of closed sets for any topology is closed under arbitrary intersections.

In[37]:= \texttt{Map[empty, SubstTest[reify, x, dif[set[top[x]], t], t -> image[inverse[IRC], fix[ACLOSURE]}]]

Out[37]= subclass[\text{image[IRC, TOPS]}, \text{fix[ACLOSURE]}] = True

In[38]:= subclass[\text{image[IRC, TOPS]}, \text{fix[ACLOSURE]}] = True

**Corollary.** Closed sets are closed under arbitrary intersections.

In[39]:= \texttt{SubstTest[implies, and\{member[x, u], subclass[u, v]\], member[x, v], \{u -> image[IRC, TOPS], v -> fix[ACLOSURE]\}] // Reverse}

Out[39]= or[equal[x, Aclosure[x]], not\{member[x, image[IRC, TOPS]]\}] = True

In[40]:= or[equal[x, Aclosure[x]], not\{member[x, image[IRC, TOPS]]\}] = True

**Theorem.** The collection of closed sets for any topology is closed under binary unions.

In[41]:= \texttt{Map[empty, SubstTest[reify, x, dif[set[top[x]], t], t -> image[inverse[IRC], binclosed[CUP]]]]}

Out[41]= subclass[\text{image[IRC, TOPS]}, \text{binclosed[CUP]}] = True

In[42]:= subclass[\text{image[IRC, TOPS]}, \text{binclosed[CUP]}] = True

**Corollary.** The class of all closed sets of any topology is closed under binary unions.

In[43]:= \texttt{SubstTest[implies, and\{member[x, u], subclass[u, v]\], member[x, v], \{u -> image[IRC, TOPS], v -> binclosed[CUP]\}] // Reverse}

Out[43]= or\{not\{member[x, image[IRC, TOPS]]\}, subclass[\text{image[CUP, cart[x, x]]}, x\] = True

In[44]:= or\{not\{member[x, image[IRC, TOPS]]\}, subclass[\text{image[CUP, cart[x, x]]}, x\] = True

**Theorem.** The collection of closed sets for any topology holds its sum class.

In[45]:= \texttt{Map[empty, SubstTest[reify, x, dif[set[top[x]], t], t -> image[inverse[IRC], fix[composite[inverse[E], IMAGE[inverse[BIGCUP]]]]]]}

Out[45]= subclass[\text{image[IRC, TOPS]}, \text{fix[composite[inverse[E], IMAGE[inverse[BIGCUP]]]]} = True

In[46]:= subclass[\text{image[IRC, TOPS]}, \text{fix[composite[inverse[E], IMAGE[inverse[BIGCUP]]]]} = True

**Corollary.** The union of all closed sets is closed.
Theorem. For any topology, the collection of closed sets holds the empty set.

Corollary. The empty set is closed for any topology.

results involving ADJOIN[0]

Although the function IRC is not an involution, the following result shows that it comes close.

Lemma.

Theorem.

Corollary. (A generalization of a result derived earlier.)
formal statement of Kelley’s Theorem 4

Lemma. A special case of an earlier result, adding back in the redundant condition $0 \in x$.

Lemma. A simplification rule.

Theorem. Variable free statement of Kelley’s Theorem 4.
Corollary. Reintroducing variables into Kelley’s Theorem 4.

In[67]:=  
SubstTest[and, subclass[u, v], subclass[v, u],  
{u \rightarrow \text{intersection[binclosed[CUP], complement[P[complement[set[0]]]]], fix[ACLOSURE],  
fix[\text{composite}[\text{inverse}[E], \text{IMAGE}[\text{inverse}[\text{BIGCUP}]]]]}, v \rightarrow \text{image[IRC, TOPS]}]}  
Out[67]=  
equal[\text{image[IRC, TOPS]},  
\text{intersection[binclosed[CUP], complement[P[complement[set[0]]]]],  
fix[ACLOSURE], fix[\text{composite}[\text{inverse}[E], \text{IMAGE}[\text{inverse}[\text{BIGCUP}]]]]} = \text{True}

In[68]:=  
\text{Map[implies[#, member[x, image[IRC, TOPS]]] \&,  
SubstTest[member, x, \text{intersection}[t, u, v, w],  
{t \rightarrow \text{binclosed[CUP]}, u \rightarrow \text{complement[P[complement[set[0]]]]}, v \rightarrow \text{fix[ACLOSURE],  
w \rightarrow \text{fix[\text{composite}[\text{inverse}[E], \text{IMAGE}[\text{inverse}[\text{BIGCUP}]]]]}}]}  
Out[68]=  
\text{or[member[x, image[IRC, TOPS]], not[equal[x, Aclosure[x]]], not[member[0, x]],  
not[member[U[x], x]], not[subclass[\text{image[CUP, cart[x, x]], x}]]} = \text{True}

In[69]:=  
\text{or[member[x_, image[IRC, TOPS]], not[equal[x_, Aclosure[x_]]], not[member[0, x_]],  
not[member[U[x_], x_]], not[subclass[\text{image[CUP, cart[x_, x_]], x_}]]} = \text{True}