CONNEX normalization rule

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In[1]:= SetDirectory["1:”; << goedel.10mar03a; << tools.m

:Package Title: goedel.10mar03a 2010 March 3 at 9:45 a.m.

It is now: 2010 Mar 3 at 12:58

Loading Simplification Rules

TOOLS.M Revised 2010 February 26

weightlimit = 40

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**summary**

A relation \( r \) is said to be connected on a class \( s \) if \( r \subseteq s \times s \subseteq r \cup \text{Id} \cup \text{inverse}[r] \). (See, for example, page 11 of the following reference.)


In this notebook a variable-free rewrite rule is derived that says that a (small) relation is connected on some set if and only if it is connected on the union of its domain and range.

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**temporary abbreviation**

It is useful to introduce the following abbreviation for the statement that \( r \) is connected on \( s \).

In[3]:= connected[r_, s_] :=
    and[subclass[r, cart[s, s]], subclass[cart[s, s], union[Id, r, inverse[r]]]]

The class of relations that are connected on some set is given by the following expression.

In[4]:= class[r, exists[s, connected[r, s]]] // InvertFix


It will be shown that this class is equal to the class CONNEX of all relations that are connected on the union of their domain and range. The union of the domain and range of a relation is often called the field of the relation, but to avoid confusion with other meanings of the word ‘field’ the abbreviation udora is used in the GOEDEL program.
Theorem. A rewrite rule that normalizes the class CONNEX.

implication in one direction

Lemma.

Lemma.

Theorem,
the reverse direction

The inclusion \( s \times s \subseteq r \cup \text{Id} \cup \text{inverse}[r] \) implies that either \( s = \text{domain}[r] \cup \text{range}[r] \) or \( s \) is a singleton. The latter possibility must be dealt with separately.

Lemma.

Observation.

Corollary.

Theorem. If \( r \) is connected on \( s \), then \( r \) is connected on \( \text{udora}[r] \).
The following lemma is needed to help with the elimination of variables.

Lemma.

Theorem. Removing the variables \( r \) and \( s \) yields an inclusion in the reverse direction.

The final step is to combine the two inclusions into an equation and make it into a rewrite rule.

Main Theorem. A relation is connected on some set if and only if it is connected on the union of its domain and range.