finite changes to a countably infinite set

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2006 August 17

summary

If one inserts a finite number of elements into a countably infinite set or if one removes a finite number of elements from a countably infinite set, the resulting set remains countably infinite.

a special case

For the special case of Omega, removing all numbers up to some point yields an infinite set.

lemmas

Lemma.
adding new elements

Cardinality is preserved for disjoint unions. The special case of disjoint unions is considered in this section. Later on, the restriction to disjoint unions will be removed.

The variable \( z \) is removed:

\[
\text{Map[implies, member[x, FINITE], equal[v, #] &,}
\text{SubstTest[class, z, or[equal[omega, u], not[equal[0, v]], not[equal[omega, w]],}
\text{not[equal[t, nat[z]]], \{t \to card[x], u \to card[union[x, y]],}
\text{v \to intersection[x, y], w \to card[y]\}], // MapNotNot // Reverse}
\]

The following rewrite rule will also be needed shortly.

The following rewrite rule will also be needed shortly.
the general case

In this section the restriction to disjointness unions will be removed. The disjointness literal is removed as follows:

\[\text{In}[16]:= \text{SubstTest}[\text{or}, \text{equal} [\text{omega}, \text{card} [\text{union} [w, y]]], \text{not} [\text{equal} [0, \text{intersection} [w, y]]], \\text{not} [\text{equal} [\text{omega}, \text{card} [y]]], \text{not} [\text{member} [w, \text{FINITE}]], w \rightarrow \text{dif} [x, y]]\]

\[\text{Out}[16]= \text{or} [\text{equal} [\text{omega}, \text{card} [\text{union} [x, y]]], \text{not} [\text{equal} [\text{omega}, \text{card} [y]]], \\text{not} [\text{member} [\text{intersection} [x, \text{complement} [y]], \text{FINITE}]], w \rightarrow \text{dif} [x, y]]\]

\[\text{In}[17]:= (\% /\cdot \{x \rightarrow x_, y \rightarrow y_\}) \rightarrow \text{Equal} \rightarrow \text{SetDelayed}\]

Theorem. The union of a finite set and a countably infinite set is countably infinite.

\[\text{In}[18]:= \text{Map} [\text{not}, \text{SubstTest} [\text{and}, \text{implies} [p_1, p_3], \text{implies} [\text{and} [p_2, p_3], p_4], \\text{not} [\text{implies} [\text{and} [p_1, p_2], p_4]], \{p_1 \rightarrow \text{member} [x, \text{FINITE}], \\text{p}2 \rightarrow \text{equal} [\text{omega}, \text{card} [y]], p_3 \rightarrow \text{member} [\text{intersection} [x, \text{complement} [y]], \text{FINITE}], p_4 \rightarrow \text{equal} [\text{omega}, \text{card} [\text{union} [x, y]]]]] \]

\[\text{Out}[18]= \text{or} [\text{equal} [\text{omega}, \text{card} [\text{union} [x, y]]], \\text{not} [\text{equal} [\text{omega}, \text{card} [y]]], \text{not} [\text{member} [x, \text{FINITE}]], w \rightarrow \text{dif} [x, y]]\]

\[\text{In}[19]:= \text{or} [\text{equal} [\text{omega}, \text{card} [\text{union} [x_, y_]]], \\text{not} [\text{equal} [\text{omega}, \text{card} [y_]]], \text{not} [\text{member} [x_, \text{FINITE}]]] := \text{True}\]

variable-free formulation

Lemma.

\[\text{In}[20]:= \text{or} [\text{and} [\text{equal} [\text{omega}, \text{card} [\text{union} [x, y]]], \text{member} [y, v]], \\text{not} [\text{equal} [\text{omega}, \text{card} [y]]], \text{not} [\text{member} [x, \text{FINITE}]]] \rightarrow \text{NotNotTest}\]

\[\text{Out}[20]= \text{or} [\text{and} [\text{equal} [\text{omega}, \text{card} [\text{union} [x, y]]], \text{member} [y, v]], \\text{not} [\text{equal} [\text{omega}, \text{card} [y]]], \text{not} [\text{member} [x, \text{FINITE}]], w \rightarrow \text{dif} [x, y]]\]

\[\text{In}[21]:= (\% /\cdot \{x \rightarrow x_, y \rightarrow y_\}) \rightarrow \text{Equal} \rightarrow \text{SetDelayed}\]

Eliminating the variables yields a statement about K-invariance. The introduction of the cover relation K here is the result of rewrite rules.

\[\text{In}[22]:= \text{Map} [\text{equal} [0, \text{composite} [\text{Id}, \text{complement} [\#]]] \&, \text{SubstTest} [\text{class}, \text{pair} [x, y]], \text{implies} [\text{and} [\text{member} [x, u], \text{member} [y, v]], \text{member} [\text{union} [x, y], v]], \{u \rightarrow \text{FINITE}, v \rightarrow \text{image} [Q, \text{set} [\text{omega}]]\}] \rightarrow \text{Reverse}\]

\[\text{Out}[22]= \text{subclass} [\text{image} [K, \text{image} [Q, \text{set} [\text{omega}]]], \text{image} [Q, \text{set} [\text{omega}]]] := \text{True}\]

\[\text{In}[23]:= \text{subclass} [\text{image} [K, \text{image} [Q, \text{set} [\text{omega}]]], \text{image} [Q, \text{set} [\text{omega}]]] := \text{True}\]
A more natural equational reformulation can be derived as a corollary:

```
In[24]:= SubstTest[and, subclass[u, v], subclass[v, u],
   {u -> image[Q, set[omega]], v -> image[CUP, cart[FINITE, image[Q, set[omega]]]]}]
```

```
```

```
In[25]:= image[CUP, cart[FINITE, image[Q, set[omega]]]] := image[Q, set[omega]]
```

---

**removing a finite number of elements**

Every subset of a countably infinite set is either finite or countably infinite.

```
In[26]:= SubstTest[implies, and(equal[omega, card[x]], subclass[z, x]),
   or(member[z, FINITE], equal[omega, card[x]]), z -> intersection[x, y]]
```

```
Out[26]= or(equal[omega, card[intersection[x, y]]],
   member[intersection[x, y], FINITE], not(equal[omega, card[x]])) = True
```

```
In[27]:= (% /. {x -> x_, y -> y_}) /. Equal -> SetDelayed
```

A union of two sets is finite if and only if both sets are finite.

```
In[28]:= Map[implies[#2, member[x, FINITE]] &,
   SubstTest[member, union[u, v], FINITE, {u -> dif[x, y], v -> y}]] // Reverse
```

```
Out[28]= or(member[x, FINITE], not(member[y, FINITE]],
   not(member[intersection[x, complement[y]], FINITE]]) = True
```

```
In[29]:= (% /. {x -> x_, y -> y_}) /. Equal -> SetDelayed
```

Theorem. Removing a finite number of elements from a countably infinite set yields a countably infinite subset.

```
In[30]:= Map[not, SubstTest[and, implies[p1, p3], implies[and[p2, p3], p4],
   implies[and[p1, p4], p5], not[implies[and[p1, p2], p5]],
   {p1 -> equal[omega, card[x]], p2 -> member[y, FINITE], p3 -> not[member[x, FINITE]],
   p4 -> not[member[dif[x, y], FINITE]], p5 -> equal[omega, card[dif[x, y]]]}]]
```

```
Out[30]= or(equal[omega, card[intersection[x, complement[y]]]],
   not(equal[omega, card[x]]), not(member[y, FINITE]])) = True
```

```
In[31]:= or(equal[omega, card[intersection[complement[y_, x_]]]],
   not(equal[omega, card[x_]]), not(member[y_, FINITE]]) := True
```

---

**variable-free formulation**

The following observation is used to derive the first lemma.
Since the empty set is finite, one obtains the following general inclusion:

\[
\text{In[32]} := \text{SubstTest}[\text{implies}, \text{subclass}[u, v], \text{subclass}[\text{image}[w, u], \text{image}[w, v]],
\{u \to \text{set}[0], v \to \text{FINITE}, w \to \text{composite}[\text{DIF}, \text{id}[\text{cart}[x, V]], \text{inverse}[\text{SECOND}]]\}]
\]

\[
\text{Out[32]} = \text{subclass}[x, \text{image}[\text{DIF}, \text{cart}[x, \text{FINITE}]]] = \text{True}
\]

\[
\text{In[33]} := \text{subclass}[x, \text{image}[\text{DIF}, \text{cart}[x, \text{FINITE}]]] := \text{True}
\]

Eliminating the variables from the theorem derived in the preceding section yields an inclusion in the opposite direction:

\[
\text{In[34]} := \text{Map}[\text{equal}[0, \text{composite}[\text{Id, complement}[\#]]] \&,
\text{SubstTest}[\text{class, pair}[x, y], \text{implies}[\text{and}[\text{member}[x, u], \text{member}[y, v]],
\text{member}[\text{dif}[x, y], u]], \{u \to \text{image}[Q, \text{set}[\text{omega}]], v \to \text{FINITE}]}] // \text{Reverse}
\]

\[
\text{Out[34]} = \text{subclass}[\text{image}[\text{DIF}, \text{cart}[\text{image}[Q, \text{set}[\text{omega}]], \text{FINITE}]], \text{image}[Q, \text{set}[\text{omega}]]] = \text{True}
\]

\[
\text{In[35]} := \% /\. \text{Equal} \to \text{SetDelayed}
\]

Combining the two inclusions yields an equation:

\[
\text{In[37]} := \text{SubstTest}[\text{and}, \text{subclass}[u, v], \text{subclass}[v, u],
\{u \to \text{image}[Q, \text{set}[\text{omega}]], v \to \text{image}[\text{DIF}, \text{cart}[\text{image}[Q, \text{set}[\text{omega}]], \text{FINITE}]]\}]
\]

\[
\text{Out[37]} = \text{True} = \text{equal}[\text{image}[\text{DIF}, \text{cart}[\text{image}[Q, \text{set}[\text{omega}]], \text{FINITE}]], \text{image}[Q, \text{set}[\text{omega}]]]
\]

\[
\text{In[38]} := \text{image}[\text{DIF}, \text{cart}[\text{image}[Q, \text{set}[\text{omega}]], \text{FINITE}]] := \text{image}[Q, \text{set}[\text{omega}]]
\]