DIV is a partial order relation

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Loading Simplification Rules

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weightlimit = 40

■ Summary

The fact that divisibility of natural numbers is a partial ordering is established in this notebook. Actually, the reflexive and transitive properties have already been established previously, so all that is left is to derive the antisymmetry property. This is done by establishing a connection between divisibility and the subclass relation, namely that, if one natural number divides another, then either the former is contained in the other, or else the latter is zero.

■ preliminary steps

Since zero requires special treatment in the derivation, it is not surprising that some simplifications involving zero are needed. These will suffice:

\[
\text{ImageComp}[\text{NATMUL, LEFT}[0], V] \quad \text{// Reverse}
\]

\[
\text{image}[\text{NATMUL, cart[singleton[0], V}}] == \text{singleton[0]}
\]

\[
\text{image}[\text{NATMUL, cart[singleton[0], V}}] := \text{singleton[0]}
\]

\[
\text{ImageComp}[\text{NATMUL, RIGHT}[0], V] \quad \text{// Reverse}
\]

\[
\text{image}[\text{NATMUL, cart[V, singleton[0]]}] == \text{singleton[0]}
\]

\[
\text{image}[\text{NATMUL, cart[V, singleton[0]]}] := \text{singleton[0]}
\]

■ a consequence of monotonicity

From the monotonicity of multiplication, we derived this special result:
By taking the union of the previous two formulas, we should get the divisibility relation:

\[
\text{implies} [\text{member} [0, x], \text{subclass} [y, \text{natzmul} [x, y]]]
\]

True

The key to transforming this result to relational form was the following step, which is also needed here:

\[
\text{intersection} [\text{composite} [\text{id} [\text{image} [V, \text{intersection} [x, \text{singleton} [0]]], \text{natzmul}, \text{left} [x]], \text{complement} [S]]] // \text{VSNormality}
\]

\[
\text{composite} [\text{id} [\text{image} [V, \text{intersection} [x, \text{singleton} [0]]], \text{intersection} [\text{complement} [S], \text{composite} [\text{natzmul}, \text{left} [x]]]]] == 0
\]

\[
\text{composite} [\text{id} [\text{image} [V, \text{intersection} [x_-, \text{singleton} [0]]], \text{intersection} [\text{complement} [S], \text{composite} [\text{natzmul}, \text{left} [x_-]]]]] := 0
\]

The idea is now to eliminate the variable \(x\) here. To do this, we need one technical lemma:

\[
\text{SubstTest} [\text{member}, \text{pair} [u, v], \text{composite} [\text{id} [\text{image} [V, \text{intersection} [x, \text{singleton} [0]]], \text{intersection} [\text{complement} [s], \text{composite} [n, \text{left} [x]]], \text{composite} [n, \text{left} [x]]], \{n -> \text{natzmul}, s -> S\}] // \text{Reverse}
\]

\[
\text{and} [\text{member} [0, x], \text{member} [u, v], \text{member} [v, V], \text{member} [x, \text{omega}], \text{member} [\text{pair} [u, v], \text{iterate} [\text{iterate} [\text{SUCC}, \text{singleton} [x]], \text{singleton} [0]]], \text{not} [\text{subclass} [u, v]]] == \text{False}
\]

\[
\text{and} [\text{member} [0, x_-], \text{member} [u_-, v], \text{member} [v_-, V], \text{member} [x_-, \text{omega}], \text{member} [\text{pair} [u_-, v_-], \text{iterate} [\text{iterate} [\text{SUCC}, \text{singleton} [x_-]], \text{singleton} [0]]], \text{not} [\text{subclass} [u_-, v_-]]] := \text{False}
\]

This technical lemma is needed to derive the following formula in which all variables have now been eliminated:

\[
\text{Map} [\text{equal} [0, \#, \&], \text{SubstTest} [\text{class}, \text{pair} [u, v], \text{exists} [x, \text{member} [\text{pair} [u, v], \text{composite} [\text{id} [\text{image} [V, \text{intersection} [x, \text{singleton} [0]]], \text{intersection} [\text{complement} [s], \text{composite} [n, \text{left} [x]]], \text{composite} [n, \text{left} [x]]], \{n -> \text{natzmul}, s -> S\}] // \text{Reverse}
\]

\[
\text{subclass} [\text{composite} [\text{natzmul}, \text{id} [\text{cart} [\text{complement} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}]], S] == \text{True}
\]

\[
\text{subclass} [\text{composite} [\text{natzmul}, \text{id} [\text{cart} [\text{complement} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}]], S] := \text{True}
\]

The next step is to eliminate \text{natzmul} from this relation to get a better formulation involving the divisibility relation. What stands in the way is the \(\text{id} [\ldots]\) factor. The complementary identity enters into this:

\[
\text{Assoc} [\text{natzmul}, \text{id} [\text{cart} [\text{omega}, V]], \text{composite} [\text{id} [\text{cart} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}]]] // \text{Reverse}
\]

\[
\text{composite} [\text{natzmul}, \text{id} [\text{cart} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}] == \text{cart} [\text{omega}, \text{singleton} [0]]
\]

\[
\text{composite} [\text{natzmul}, \text{id} [\text{cart} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}] := \text{cart} [\text{omega}, \text{singleton} [0]]]
\]

By taking the union of the previous two formulas, we should get the divisibility relation:

\[
\text{SubstTest} [\text{composite}, \text{natzmul}, \text{union} [u, v], \text{inverse} [\text{SECOND}], \{u -> \text{id} [\text{cart} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], v -> \text{id} [\text{cart} [\text{complement} [\text{P} [\text{complement} [\text{singleton} [0]]], V]]]] // \text{Reverse}
\]

\[
\text{union} [\text{cart} [\text{omega}, \text{singleton} [0]], \text{composite} [\text{natzmul}, \text{id} [\text{cart} [\text{complement} [\text{P} [\text{complement} [\text{singleton} [0]]], V]], \text{inverse} [\text{SECOND}]]] == \text{DIV}
\]
union[cart[omega, singleton[0]], composite[NATMUL, id[cart[complement[P[complement[singleton[0]]], V]], inverse[SECOND]]]] := DIV

The final step is this:

SubstTest[intersection, union[u, v], w, 
  {u -> composite[NATMUL, id[cart[complement[P[complement[singleton[0]]], V]], 
      inverse[SECOND]], v -> cart[omega, singleton[0]], w -> complement[S]]}]

intersection[DIV, complement[S]] ==
cart[intersection[omega, complement[singleton[0]]], singleton[0]]

intersection[DIV, complement[S]] :=
cart[intersection[omega, complement[singleton[0]]], singleton[0]]

There is a companion formula involving the inverses:

intersection[inverse[DIV], complement[inverse[S]]] // DoubleInverse

intersection[complement[inverse[S]], inverse[DIV]] ==
cart[singleton[0], intersection[omega, complement[singleton[0]]]]

intersection[complement[inverse[S]], inverse[DIV]] :=
cart[singleton[0], intersection[omega, complement[singleton[0]]]]

■ final steps

First we simply transpose the complements of S and inverse[S] in the above formulas to the other side of the inclusion.

SubstTest[subclass, intersection[z, complement[S]],
  cart[omega, singleton[0]], z -> DIV] // Reverse

subclass[DIV, union[S, cart[omega, singleton[0]]]] := True

subclass[DIV, union[S, cart[omega, singleton[0]]]] := True

SubstTest[subclass, inverse[u], inverse[v],
  {u -> DIV, v -> union[S, cart[omega, singleton[0]]]}]

subclass[inverse[DIV], union[cart[singleton[0], omega], inverse[S]]] := True

subclass[inverse[DIV], union[cart[singleton[0], omega], inverse[S]]] := True

Next we use the fact that the diversity relation Di is the union of complement[S] and complement[inverse[S]], which allows us to combine the two formulas into a single one.

SubstTest[intersection, DIV, union[u, v], inverse[DIV],
  {u -> complement[S], v -> complement[inverse[S]]}]

intersection[Di, DIV, inverse[DIV]] == 0

intersection[Di, DIV, inverse[DIV]] := 0

Again, we transpose Di to the other side of the inclusion, where it becomes its complement, the identity relation Id.
The final steps are to restrict to \( \omega \) and use the (known) reverse inclusion.

\[
\text{SubstTest[subclass, intersection[DIV, inverse[DIV]], intersection[u, v], }
\{u -> \text{id}, v -> \text{cart[omega, omega]}\}]
\text{subclass[intersection[DIV, inverse[DIV]], id[omega]] == True}
\]
\[
\text{subclass[intersection[DIV, inverse[DIV]], id[omega]] := True}
\]
\[
\text{SubstTest[and, subclass[u, v], subclass[v, u], }
\{u -> \text{id[omega]}, v -> \text{intersection[DIV, inverse[DIV]]}\}]
\text{True == equal[\text{id[omega]}, \text{intersection[DIV, inverse[DIV]]}]}
\]

This is the final form of the antisymmetry property.

\[
\text{intersection[DIV, inverse[DIV]] := id[omega]}
\]

As a corollary it follows that \( \text{DIV} \) belongs to the class \( \text{PO} \) of all (small) partial orderings.

\[
\text{member[DIV, PO]}
\]

\text{True}