domain and range of rational numbers

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In[1]:= SetDirectory["l:"]; << goedel.12jun03b

:Package Title: goedel.1jun03b 2012 June 3 at 7:55 p.m.

Loading takes about seventeen minutes, half that time due to builtin pauses.

It is now: 2012 Jun 8 at 11:54

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now: 2012 Jun 8 at 12:11

summary

In the GOEDEL program, rational numbers have been defined as certain functions whose graphs are straight lines through the origin in the integer plane $\mathbb{Z} \times \mathbb{Z}$. The rational numbers themselves are ranges of subgroups of the direct product of the integer addition group INTADD with itself. Consequently, the the domain and range of any rational number are ranges of subgroups of INTADD.

ranges of rationals

The term group in the GOEDEL program refers to an associative composition law, and the underlying set on which this acts is its range. Accordingly the binary operation INTADD of integer addition is called a group, and the range of this group is the set $\mathbb{Z}$ of integers.

Lemma. If $x$ is the range of a subgroup of the direct product of INTADD with itself, then range[x] is the range of a subgroup of the group INTADD.

In[2]:= SubstTest[implies, and[member[u, GROUPS], member[v, GROUPS],
member[x, image[IMAGE[SECOND], intersection[GROUPS, P[direct[u, v]]]]],
member[range[x], image[IMAGE[SECOND], intersection[GROUPS, P[v]]]],
{u \rightarrow INTADD, v \rightarrow INTADD}] // Reverse

Out[2]= or[member[range[x], image[VERTSECT[INTDIV], Z]], not[member[x, image[IMAGE[SECOND],
intersection[GROUPS, P[composite[cross[INTADD, INTADD], TWIST]]]]]]] = True
Every subgroup of integer addition is a cyclic subgroup. Thus the range of any subgroup of INTADD is the set of multiples of some integer. These are just the nonempty vertical sections of the integer divisibility relation INTDIV.

Theorem. The range of a rational number is a nonempty vertical section of INTDIV.

Eliminating the variable \( x \) yields the following inclusion.

Corollary.

Lemma. The opposite inclusion also holds.

Combining the two inclusions yields an equation that can be made into a rewrite rule.

Theorem. A formula for the class of ranges of rational numbers.
domains of fractions

The domains of rationals, like their ranges, are also vertical sections of \texttt{INTDIV}, but with the extra wrinkle that one has to take into account that the denominator of a fraction can not be zero. The vertical straight line through the origin in the plane \( \mathbb{Z} \times \mathbb{Z} \) is not a function, and thus not a rational number. Thus the domain of a rational number can not the range of the trivial one-element subgroup of \texttt{INTADD}.

Lemma. If \( x \) is the range of a subgroup of the direct product of \texttt{INTADD} with itself, then \texttt{domain[x]} is the range of a subgroup of the group \texttt{INTADD}.

\begin{align*}
\text{In[12]} &= \text{SubstTest[implies, and\{member[u, GROUPS], member[v, GROUPS], member[x, image[IMAGE[SECOND], intersection\{GROUPS, P[direct[u, v]]\}]], member[domain[x], image[IMAGE[SECOND], intersection\{GROUPS, P[u]\}]], \{u \rightarrow \text{INTADD}, v \rightarrow \text{INTADD}\}]} // \text{Reverse} \\
\text{Out[12]} &= \text{or\{member[domain[x], image[VERTSECT[INTDIV], \mathbb{Z}]], not\{member[x, image[IMAGE[SECOND], intersection\{GROUPS, P[\text{composite}[cross[\text{INTADD}, \text{INTADD}], \text{TWIST}]]\}]]\} \Rightarrow True} \\
\text{In[13]} &= \text{or\{member[domain[x_], image[VERTSECT[INTDIV], \mathbb{Z}]], not\{member[x_, image[IMAGE[SECOND], intersection\{GROUPS, P[\text{composite}[cross[\text{INTADD}, \text{INTADD}], \text{TWIST}]]\}]]\} := True}
\end{align*}

Theorem. The domain of a rational number is a nonempty vertical section of \texttt{INTADD}.

\begin{align*}
\text{In[14]} &= \text{Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3], not\{implies[p1, p3]\}], \{p1 \rightarrow \text{member}[x, \text{RATS}], p2 \rightarrow \text{member}[x, image[IMAGE[SECOND], intersection\{GROUPS, P[\text{direct[\text{INTADD}, \text{INTADD}]]}\]], p3 \rightarrow \text{member[domain[x], image[VERTSECT[INTDIV], \mathbb{Z}]]}\}]} // \text{Reverse} \\
\text{Out[14]} &= \text{or\{member[domain[x], image[VERTSECT[INTDIV], \mathbb{Z}]], not\{member[x, \text{RATS}]\} \Rightarrow True} \\
\text{In[15]} &= \text{or\{member[domain[x_], image[VERTSECT[INTDIV], \mathbb{Z}]], not\{member[x_, \text{RATS}]\} := True}
\end{align*}

Corollary. Eliminating the variable \( x \) yields an inclusion.

\begin{align*}
\text{In[16]} &= \text{Map[equal[V, domain[\#]] \&, SubstTest[reify, x, case\{implies\{member[x, y], member[domain[x], z]\}], y \rightarrow \text{RATS}, z \rightarrow image[VERTSECT[INTDIV], \mathbb{Z}]\}]} \\
\text{Out[16]} &= \text{subclass[\{image[IMAGE[\text{FIRST}], \text{RATS}], image[VERTSECT[INTDIV], \mathbb{Z}]\} \Rightarrow True} \\
\text{In[17]} &= \% \rightarrow \text{SetDelayed}
\end{align*}

Theorem. If \( x \) is a non-zero integer, then \texttt{inverse[inttimes[x]]} is a fraction.

\begin{align*}
\text{In[18]} &= \text{SubstTest[implies, and\{member[x, \mathbb{Z}], member[y, \mathbb{Z}], not\{equal[x, id[\text{omega}]]\}, member[composite[\text{inverse}[inttimes[x]], inttimes[y]], \text{RATS}], y \rightarrow \text{plus}[\text{set[0]]}]} // \text{Reverse} \\
\text{Out[18]} &= \text{or\{equal[x, id[\text{omega}]], member[\text{inverse}[\text{inttimes}[x]], \text{RATS}], not\{member[x, \mathbb{Z}]\} \Rightarrow True}
\end{align*}
Theorem. One of the nonempty vertical sections of INTDIV is the singleton of the integer zero (= id[ω]).

Theorem. There is no rational number with whose domain is \{id[ω]\}.

Lemma. (If \( t \) is a member of the singleton \( x \), then \( x = \{t\} \).)

Theorem. There is no rational number with whose domain is \{id[ω]\}.

Corollary.

Lemma.
Theorem. Every other vertical section of \textsc{INTDIV} is the domain of some rational number.

\begin{verbatim}
In[30]:= Map[not, SubstTest[and, implies[p1, p2], implies[p2, p3],
    not[implies[p1, p3]], {p1 \rightarrow and[member[u, Z], not[equal[u, id[omega]]]]},
    p2 \rightarrow member[inverse[inttimes[u]], RATS],
    p3 \rightarrow member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]]]]) // Reverse
Out[30]= or[equal[u, id[omega]],
    member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]], not[member[u, Z]]] = True
\end{verbatim}

Lemma.

\begin{verbatim}
In[32]:= Map[equal[domain[#], V] &,
    SubstTest[reify, u, case[or[member[image[INTDIV, set[u]], image[IMAGE[FIRST], RATS]],
    not[member[u, v]]], {v \rightarrow dif[Z, set[id[omega]]]}]]]
    image[inverse[VERTSECT[INTDIV]], image[IMAGE[FIRST], RATS]], set[id[omega]]]] = True
\end{verbatim}

Corollary.

\begin{verbatim}
In[34]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
    {t \rightarrow VERTSECT[INTDIV], u \rightarrow Z, v \rightarrow union[image[inverse[VERTSECT[INTDIV]],
    image[IMAGE[FIRST], RATS]], set[id[omega]]]]}] // Reverse
Out[34]= subclass[image[VERTSECT[INTDIV], Z],
    union[image[IMAGE[FIRST], RATS], set[set[id[omega]]]]] = True
\end{verbatim}

Theorem. A formula for the set of domains of rational numbers.

\begin{verbatim}
In[36]:= SubstTest[and, subclass[u, v], subclass[v, u], {u \rightarrow image[IMAGE[FIRST], RATS],
    v \rightarrow dif[image[VERTSECT[INTDIV], Z], set[set[id[omega]]]]}]]
Out[36]= equal[image[IMAGE[FIRST], RATS],
    intersection[complement[set[set[id[omega]]]], image[VERTSECT[INTDIV], Z]]] = True
In[37]:= image[IMAGE[FIRST], RATS] :=
    intersection[complement[set[set[id[omega]]]], image[VERTSECT[INTDIV], Z]]
\end{verbatim}