finite rank

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In[1]:= << goedel54.01a; << tools.m

It is now: 2004 Feb 4 at 14:21

Loading Simplification Rules

TOOLS.M Revised 2004 January 3

weightlimit = 40

summary

A set has finite rank if and only if it is regular and hereditarily finite. Three steps are used to derive this fact. In the first section below it is shown that the rank of the set of all regular hereditarily finite sets is \( \omega \). In the second section it is shown that the set of the ranks of the elements of this set is again \( \omega \). The theorem itself is obtained in the third section. In each section a variable–free statement is derived.

the set of regular hereditarily finite sets

Although the class \texttt{FINITE} of all finite sets is a proper class, the class of regular hereditarily finite sets is a set.

\begin{verbatim}
In[2]:= SubstTest[implies, member[x, REGULAR], member[image[ZN, x], V], x -> omega]


In[3]:= member[intersection[REGULAR, H[FINITE]], V] := True
\end{verbatim}

The first task will be to show that the rank of this set is \( \omega \). It is easy to obtain a lower bound:

\begin{verbatim}
In[4]:= SubstTest[implies, subclass[u, v], subclass[rank[u], rank[v]],
   {u -> omega, v -> intersection[REGULAR, H[FINITE]]}]


In[5]:= subclass[omega, rank[intersection[REGULAR, H[FINITE]]]] := True
\end{verbatim}

The following lemmas are needed to obtain an upper bound.

\begin{verbatim}
In[6]:= SubstTest[member, pair[x, y], composite[Id, x], z -> ZN] // Reverse

Out[6]= and[member[x, V], member[y, V], member[pair[x, y], ZN]] = member[pair[x, y], ZN]

In[7]:= and[member[x, V], member[y, V], member[pair[x, y], ZN]] := member[pair[x, y], ZN]
\end{verbatim}
Lemma.

In[8]:= SubstTest[implies, member[y, z], subclass[A[z], y],
   z -> intersection[OMEGA, image[inverse[ZN], singleton[x]]]]

Out[8]= or[not[member[y, OMEGA]], not[member[pair[y, x], ZN]], subclass[rank[x], y]] = True

In[9]:= or[not[member[y, OMEGA]], not[member[pair[y, x], ZN]], subclass[rank[x], y]] := True

Lemma.

In[10]:= Map[implies[#, member[pair[x, y], ZN]] &,
   SubstTest[member, y, image[z, singleton[x]], z -> ZN] /. y -> image[ZN, x]
Out[10]= or[member[pair[x, image[ZN, x]], ZN],
   not[member[z, V]], not[member[image[ZN, x], V]]] = True

In[11]:= (% /. x -> x_) /. Equal -> SetDelayed

Lemma.

In[12]:= SubstTest[or, member[pair[x, image[ZN, x]], ZN],
   not[member[x, V]], not[member[image[ZN, x], V]],
   x -> omega]

In[13]:= member[pair[omega, intersection[REGULAR, H[FINITE]]], ZN] := True

This yields an upper bound for the rank of the set of regular hereditarily finite sets.

In[14]:= SubstTest[implies,
   and[member[pair[x, y], ZN], member[x, OMEGA]], subclass[rank[y], x],
   {x -> omega, y -> intersection[REGULAR, H[FINITE]]}]

In[15]:= subclass[rank[intersection[REGULAR, H[FINITE]]], omega] := True

Putting together the two inclusions yields the result that the rank of the set of regular hereditarily finite sets is omega.

In[16]:= SubstTest[and, subclass[u, v], subclass[v, u],
   {u -> omega, v -> rank[intersection[REGULAR, H[FINITE]]]}]
Out[16]= True = equal[omega, rank[intersection[REGULAR, H[FINITE]]]]

In[17]:= rank[intersection[REGULAR, H[FINITE]]] := omega

ranks of regular hereditarily finite sets

In this section it will be shown that the set of the ranks of regular hereditarily finite sets is omega. Every natural number is its own rank:

In[18]:= ImageComp[RANK, id[OMEGA], omega] // Reverse
Out[18]= image[RANK, omega] = omega
Since every natural number is a regular hereditarily finite set, it follows that \( \omega \) is contained in the set of ranks of hereditarily finite sets.

The result of the preceding section implies the opposite inclusion:

Putting together these two inclusions yields the following equation:

This lemma provides an inclusion in one direction:

The opposite inclusion also holds:

In the present section it is shown that there are no other sets with finite rank. The following lemma is needed:

This lemma provides an inclusion in one direction:

The opposite inclusion also holds:
These two inclusions can be combined into an equation:

\[ \text{SubstTest}[\text{and}, \text{subclass}[u, v], \text{subclass}[v, u], \\
{u \rightarrow \text{intersection[REGULAR, H[FINITE]]}, v \rightarrow \text{image[\text{inverse[RANK], omega}]}]} \]

This result can be expressed as follows, making the finiteness property more explicit:

\[ \text{image[\text{inverse[RANK], omega}]} := \text{intersection[REGULAR, H[FINITE]]} \]