nonempty finite partial orders have maximal elements

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summary

Any nonempty finite partial order has a maximal element. One can extend a finite partial order $p$ to a total order, and then one can cut that down to a total order $t$ with the properties that $p \subseteq t$ and $\text{fix}(p) = \text{fix}(t)$. If $p$ is not empty, then $t$ is a non-empty finite total order, and so has a $t$-greatest element which is a maximal element for $p$.

derivation

Several wrappers are used to reduce the number of literals.

Lemma. If any greatest element of a nonempty total order is a maximal element of a partial order, then the partial order has a maximal element.

Lemma. An inclusion for $\text{funpart}$. 

\[ \text{SubstTest}[\text{implies, subclass}[p, t], \\
\text{subclass}[	ext{intersection}[\text{domain}[p], \text{domain}[\text{funpart}[t]]], \text{domain}[\text{funpart}[p]]], \\
\{p \rightarrow \text{po}[x], t \rightarrow \text{to}[\text{fin}[y]]\}] // \text{Reverse} \]

\[ \text{Out}[6]= \text{or}[\text{not}[	ext{subclass}[\text{po}[x], \text{to}[\text{fin}[y]]]], \text{subclass}[	ext{\text{intersection}[\text{fix}[	ext{funpart}[\text{to}[\text{fin}[y]]]], \text{fix}[\text{po}[x]]], \text{fix}[	ext{funpart}[\text{po}[x]]]]] = \text{True} \]
Lemma. If a nonempty partial order can be extended to a finite total order with the same fixed-point class, then the partial order has a maximal member.

Corollary. All of the wrappers can be removed.

Corollary. The hypothesis \( \text{fix}[x] = \text{fix}[y] \) can be omitted and the finiteness hypothesis transferred to \( x \). Some lemmas are needed.
Any finite partial order can be extended to a total order. A variable-free statement of this fact is available in the Theorem.

Comment: To shorten execution time, these proof steps are omitted in the following theorem: implies[and[p0, p1], p3], implies[and[p0, p2], p4] and implies[and[p0, p2], p5].

Theorem. If a nonempty finite partial order \( x \) is contained in a total order \( y \), then it has a maximal element.

The variable \( y \) can be eliminated.

Any finite partial order can be extended to a total order. A variable-free statement of this fact is available in the GOEDEL program. The following theorem just introduces a variable to facilitate reasoning.
Theorem. Any nonempty finite partial order has a maximal element.

In[24]:= Map[not, SubstTest[and, implies[p1, p3],
  implies[and[p1, p2, p3], p4],
  {p1 -> member[x, FINITE], PARTIALORDER[x]}, p2 -> empty[funpart[x]],
  p3 -> member[x, image[inverse[S], TO]], p4 -> empty[x]]] // Reverse

Out[24]= or[equal[0, x], not[equal[0, funpart[x]]],
  not[member[x, FINITE]], not[PARTIALORDER[x]]] = True

In[25]:= or[equal[0, x_], not[equal[0, funpart[x_]]],
  not[member[x_, FINITE]], not[PARTIALORDER[x_]]] := True

Corollary. Restatement with wrappers.

In[26]:= SubstTest[or, equal[0, t], not[equal[0, funpart[t]]],
  not[member[t, FINITE]], not[PARTIALORDER[t]], t -> po[fin[x]]] // Reverse

Out[26]= or[equal[0, po[fin[x]]], not[equal[0, funpart[po[fin[x]]]]]] = True

In[27]:= (% /. x -> x_) /. Equal -> SetDelayed

A better rewrite rule is possible.

Theorem.

In[28]:= equiv[equal[0, funpart[po[fin[x]]]], equal[0, po[fin[x]]]]

Out[28]= True

In[30]:= equal[0, funpart[po[fin[x_]]]] := equal[0, po[fin[x]]]

A variable-free statement can also be derived.

In[31]:= subclass[intersection[FINE, PO, image[inverse[FUNPART], set[0]]], set[0]] // AssertTest

Out[31]= subclass[intersection[FINE, PO, image[inverse[FUNPART], set[0]]], set[0]] = True

In[32]:= % /. Equal -> SetDelayed

This can be strengthened to an equation and made into a rewrite rule.

In[33]:= equal[intersection[FINE, PO, image[inverse[FUNPART], set[0]]], set[0]]

Out[33]= True

In[34]:= intersection[FINE, PO, image[inverse[FUNPART], set[0]]] := set[0]