summary

The fixed point set of any small total order is equipollent to a chain of the subset relation. This fact is already known when \texttt{axch} holds. In that case any set is the fixed point sets of a small total order and any set is equipollent to an ordinal, which is a chain of the subset relation. In this notebook the axiom of choice is not assumed to be true.

rewrite rules for SIMILAR

Theorem.

\begin{align*}
\text{In[2]} &= \text{SubstTest}[\text{implies, and, member[pair[u, v], composite[Id, t]], member[u, x]],} \\
&\quad \text{member[v, image[t, x]], } t \rightarrow \text{SIMILAR} \] // Reverse

\text{Out[2]} &= \text{or[member[v, image[SIMILAR, x]],} \\
&\quad \text{not[member[u, x]], not[member[pair[u, v], SIMILAR]]}] := \text{True}

\text{In[3]} &= \text{or[member[v, image[SIMILAR, x]],} \\
&\quad \text{not[member[u, x]], not[member[pair[u, v], SIMILAR]]}] := \text{True}
\end{align*}

Theorem.
total orders

Theorem. Any set similar to a total order is a total order.

```
In[6]:= SubstTest[implies, and[member[pair[u, v], composite[Id, x]], member[u, image[x, y]],
                          member[v, image[x, y]], {x -> SIMILAR, y -> TO}] // MapNotNot // Reverse
Out[6]= or[not[member[pair[u, v], SIMILAR]], not[TOTALORDER[u]], TOTALORDER[v]] := True
```

Theorem. Any set similar to a total order is a total order.

```
In[7]:= or[not[member[pair[u, v], SIMILAR]], not[TOTALORDER[u]], TOTALORDER[v]] := True
```

Lemma.

```
In[8]:= SubstTest[or, member[u, image[SIMILAR, x]],
              not[member[v, x]], not[member[pair[u, v], SIMILAR]],
              x -> intersection[TO, image[IMAGE[Id[S]], image[CART, Id]]]] // Reverse // MapNotNot
Out[8]= or[member[u, image[SIMILAR, intersection[TO, P[S]]]],
           not[member[pair[u, v], SIMILAR]], not[subclass[v, S]], not[TOTALORDER[v]]] := True
```

Lemma.
Lemma. (Eliminate the variables \( u \) and \( v \).)

\[
\text{Map[empty[composite[Id, complement[#]]] \&, SubstTest[class, pair[u, v],}
\text{ implies[and[member[pair[u, v], s], member[u, t], member[v, r]], member[u, z]],}
\text{ \{x \to image[IMAGE[id[S]], image[CART, Id]], s \to SIMILAR, t \to TO,}
\text{ z \to image[SIMILAR, intersection[TO, image[IMAGE[id[S]], image[CART, Id]]]]\}]]}]
\]

\[
\text{subclass[TO, image[SIMILAR, intersection[TO, P[S]]]] = True}
\]

\[
% /. \{u \to u\_, v \to v\_\} /. \text{Equal} \to \text{SetDelayed}
\]

Lemma. Every total ordering is similar to a total ordering by inclusion.

\[
\text{SubstTest[and, subclass[u, v], subclass[v, u],}
\text{ \{u \to TO, v \to image[SIMILAR, intersection[TO, P[S]]]\}]}\]

\[
\text{equal[TO, image[SIMILAR, intersection[TO, P[S]]]] = True}
\]

\[
\text{image[SIMILAR, intersection[TO, P[S]]]} = \text{TO}
\]

**One-one transforms of total orders**

The following rewrite rule was derived 2008 May 17 but was not added to the GOEDEL program at that time. It is rederived here.

Theorem. Application to one-to-one transforms of total orders.
The application of \texttt{reify} done in 2008 needs to be modified slightly, but the final result is still the same.

Lemma. (Eliminating the \texttt{oopart} wrapper.)

\begin{verbatim}
In[22]:= SubstTest[implies, equal[x, oopart[t]],
       invariant[IMAGE[x], image[FIX, TO]], t \to x] // Reverse
Out[22]= or[not[FUNCTION[x]], not[FUNCTION[inverse[x]]],
      subclass[image[IMAGE[x]], image[IMAGE[inverse[DUP]], TO]],
      image[IMAGE[inverse[DUP]], TO]] \Rightarrow True
\end{verbatim}

\begin{verbatim}
In[23]:= (\% / . x \to x _) /. Equal \to SetDelayed
\end{verbatim}

The next step eliminates the variable \(x\) by using \texttt{reify} and \texttt{case}.

Lemma.

\begin{verbatim}
In[24]:= Map[equal[V, domain[#]] &,
      SubstTest[reify, x, case[implies[subclass[P[x], u], invariant[IMAGE[x], v]]],
      \{u \to BIJ, v \to image[FIX, TO]\}]]
Out[24]= subclass[image[Q, image[IMAGE[inverse[DUP]], TO]],
       image[IMAGE[inverse[DUP]], TO]] \Rightarrow True
\end{verbatim}

\begin{verbatim}
In[25]:= \% / . Equal \to SetDelayed
\end{verbatim}

The inclusion in the lemma can be strengthened to an equation that can be made into a rewrite rule.

Theorem. Any set equipollent to the fixed point set of a total ordering is the fixed point set of a total ordering,

\begin{verbatim}
In[26]:= SubstTest[and, subclass[u, v], subclass[v, u],
      \{u \to image[Q, image[IMAGE[inverse[DUP]], TO]], v \to image[IMAGE[inverse[DUP]], TO]\}]
Out[26]= equal[image[Q, image[IMAGE[inverse[DUP]], TO]], image[IMAGE[inverse[DUP]], TO]] \Rightarrow True
\end{verbatim}

\begin{verbatim}
In[27]:= image[Q, image[IMAGE[inverse[DUP]], TO]] := image[IMAGE[inverse[DUP]], TO]
\end{verbatim}

\textbf{main theorem}

In this section it is shown that the fixed point set of any small total ordering is equipollent to a chain of the subset relation.
Theorem. (An inclusion in one direction.)

```
In[28]:= SubstTest[implies, subclass[u, v], subclass[image[t, u], image[t, v]],
   {t -> Q, u -> chains[S], v -> image[IMAGE[inverse[DUP]], TO]}] // Reverse
Out[28]= subclass[image[Q, chains[S]], image[IMAGE[inverse[DUP]], TO]] = True
```

In[29]:= % /. Equal -> SetDelayed

Theorem. Similar relations have equipollent fixed point sets.

```
In[30]:= SubstTest[implies, subclass[u, v], subclass[composite[t, u], composite[t, v]],
   {t -> composite[IMAGE[inverse[DUP]], SIMILAR],
    u -> Id, v -> composite[inverse[FIX], FIX]}] // Reverse
Out[30]= subclass[composite[IMAGE[inverse[DUP]], SIMILAR],
    composite[Q, IMAGE[inverse[DUP]]]] = True
```

```
In[31]:= subclass[composite[IMAGE[inverse[DUP]], SIMILAR],
    composite[Q, IMAGE[inverse[DUP]]]] := True
```

Theorem. The fixed point set for a total ordering by inclusion is a chain of the subset relation.

```
In[32]:= ImageComp[FIX, IMAGE[id[S]]], image[CART, id[chains[S]]]]] // Reverse
Out[32]= image[IMAGE[inverse[DUP]], intersection[TO, P[S]]] = chains[S]
```

```
In[33]:= image[IMAGE[inverse[DUP]], intersection[TO, P[S]]] := chains[S]
```

Lemma. An inclusion in the opposite direction.

```
In[34]:= SubstTest[implies, subclass[u, v], subclass[image[u, w], image[v, w]],
   {u -> composite[IMAGE[inverse[DUP]], SIMILAR],
    v -> composite[Q, IMAGE[inverse[DUP]]], w -> intersection[TO, P[S]]}] // Reverse
Out[34]= subclass[image[IMAGE[inverse[DUP]], TO], image[Q, chains[S]]] = True
```

In[35]:= % /. Equal -> SetDelayed

Comment. It is not clear how best to orient the following rewrite rule.

Main Theorem. The fixed point set of any small total ordering is equipollent to a chain of the subset relation.

```
In[36]:= SubstTest[and, subclass[u, v], subclass[v, u],
   {u -> image[IMAGE[inverse[DUP]], TO], v -> image[Q, chains[S]]})
Out[36]= equal[image[Q, chains[S]], image[IMAGE[inverse[DUP]], TO]] = True
```

```
In[37]:= image[Q, chains[S]] := image[IMAGE[inverse[DUP]], TO]
```

Corollary.
In[38]:= SubstTest[implies, subclass[u, v],
        subclass[image[t, u], image[t, v]], {t → Q, u → OMEGA, v → chains[S]}] // Reverse

Out[38]= subclass[image[Q, OMEGA], image[IMAGE[inverse[DUP]], TO]] == True

In[39]:= subclass[image[Q, OMEGA], image[IMAGE[inverse[DUP]], TO]] := True

Comment. If follows from this result that when \texttt{axch} holds, \texttt{image[Q, Ω]} = V, so in that case \texttt{image[FIX, TO]} = V, a result already known. It is unclear from what has been derived here whether the statement that any set can be totally ordered is weaker than the axiom of choice.