images of subgroup ranges under homomorphisms

Johan G. F. Belinfante
2012 February 14

\[ \text{In[1]} := \text{SetDirectory["l:"]; << goedel.12feb14a} \]

<table>
<thead>
<tr>
<th>:Package Title: goedel.12feb14a</th>
<th>2012 February 14 at 1:50 p.m.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loading takes about thirteen minutes, half that time due to built-in pauses.</td>
<td></td>
</tr>
<tr>
<td>It is now: 2012 Feb 14 at 15:35</td>
<td></td>
</tr>
<tr>
<td>Loading Simplification Rules</td>
<td></td>
</tr>
<tr>
<td>TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3</td>
<td></td>
</tr>
<tr>
<td>weightlimit = 40</td>
<td></td>
</tr>
<tr>
<td>Loading completed.</td>
<td></td>
</tr>
<tr>
<td>It is now: 2012 Feb 14 at 15:49</td>
<td></td>
</tr>
</tbody>
</table>

---

**summary**

The image of the range of a subgroup of a group \( x \) under a binary homomorphism from \( x \) to a group \( y \) is the range of a subgroup of \( y \).

---

**derivation**

Lemma.

\[ \text{In[5]} := \text{SubstTest[implies, and\{member[v, GROUPS], member[y, GROUPS], member[u, binhom[v, y]], member[range[u], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]], \{u \rightarrow \text{composite[t, id[w]]}, v \rightarrow \text{composite[x, id[cart[w, w]]]}} \}} \]

\[ \text{Out[5]} = \text{or\{member[image[t, w], image[IMAGE[SECOND], intersection[GROUPS, P[y]]]], not\{member[y, GROUPS]\], not\{member[composite[t, id[w]], binhom[composite[x, id[cart[w, w]], y]]\], not\{member[composite[x, id[cart[w, w]], GROUPS]\] = True} \]

\[ \text{In[6]} := \text{(}/.\{t \rightarrow \text{t_}, w \rightarrow \text{w_}, x \rightarrow \text{x_}, y \rightarrow \text{y_}\} /\text{. Equal} \rightarrow \text{SetDelayed}}} \]

Theorem. If \( t \) is a binary homomorphism from a group \( x \) to a group \( y \) and if \( w \) is the range of a subgroup of \( x \), then \( \text{image}[t, w] \) is the range of a subgroup of \( y \).
Lemma. The variable w can be eliminated. The funpart wrapper is used to convert an inverse image to a direct image.

Lemma. Eliminate the funpart wrapper. (This introduces a redundant literal.)

Finally, the redundant literal FUNCTION[t] is eliminated in the usual fashion.

Theorem. If t is a binary homomorphism from a group x to a group y, then IMAGE[t] conducts the class of ranges of subgroups of x to the class of ranges of subgroups of y.
In[25]:= SubstTest[and, implies[p, q], or[p, q], {p \rightarrow FUNCTION[t],
q \rightarrow or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]],
not[member[y, GROUPS]],
subclass[image[IMAGE[t]], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]]}

Out[25]= or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]],
not[member[y, GROUPS]],
subclass[image[IMAGE[t]], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
image[IMAGE[SECOND], intersection[GROUPS, P[y]]]] = True

In[27]:= or[not[member[t_, binhom[x_, y_]]], not[member[x_, GROUPS]], not[member[y_, GROUPS]],
subclass[image[IMAGE[t_], image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]],
image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]]] := True

The variable t can also be eliminated.

Theorem. Images of ranges of subgroups of a group x under binary homomorphisms from x to a group y are ranges of subgroups of y.

In[30]:= Map[equal[V, domain[#]] &,
SubstTest[reify, t, case[or[not[member[t, binhom[x, y]]], not[member[x, GROUPS]],
not[member[y, GROUPS]], subclass[image[IMAGE[t, u], v]],
{u \rightarrow image[IMAGE[SECOND], intersection[GROUPS, P[x]]]],
v \rightarrow image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]]]}

Out[30]= or[not[member[x, GROUPS]], not[member[y, GROUPS]], subclass[
image[IMG, cart[binhom[x, y], image[IMAGE[SECOND], intersection[GROUPS, P[x]]]]],
image[IMAGE[SECOND], intersection[GROUPS, P[y]]]]] = True

In[32]:= or[not[member[x_, GROUPS]], not[member[y_, GROUPS]], subclass[
image[IMG, cart[binhom[x_, y_], image[IMAGE[SECOND], intersection[GROUPS, P[x_]]]]],
image[IMAGE[SECOND], intersection[GROUPS, P[y_]]]]] := True