immediate successor

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In[1]:= SetDirectory["1:"]; << goedel.09oct09a; << tools.m

It is now: 2009 Oct 13 at 16:21

Loading Simplification Rules

TOOLS.M  Revised 2009 September 15

weightlimit = 40

summary

Patrick Suppes defines the concept of immediate successor for an arbitrary relation \( r \) as follows. An element \( y \) is an \( r \)-immediate successor of \( x \) if \( x \) is \( r \)-related to \( y \) and for all \( z \), if \( x \) is \( r \)-related to \( z \), then either \( y = z \) or \( y \) is \( r \)-related to \( z \).

In[2]:= "Patrick Suppes, Axiomatic Set Theory, Dover Publications, Inc.,

In this notebook it is shown that the immediate successor relation corresponding to the strict part of a total order relation is the cover relation of the total order relation.

immediate successor relation

The immediate successor relation for a relation \( r \) is the following:

In[3]:= class[pair[x, y], and[member[pair[x, y], r],
     forall[z, implies[member[pair[x, z], r], or[equal[y, z], member[pair[y, z], r]]]]]]}

Out[3]= composite[Id,
   intersection[r, complement[composite[intersection[Di, complement[inverse[r]], r]], r]]]

For convenience, the following temporary abbreviation will be introduced:

In[4]:= immediatesuccessor[r_] :=
   intersection[r, complement[composite[intersection[Di, complement[inverse[r]], r]]]

Imm-succ.nb 1
strict part of a total order relation

Theorem.

In[5]:= SubstTest[subclass, range[funpart[t]], range[t], t \[Function] intersection[Di, to[x]]] // Reverse
Out[5]= subclass[range[funpart[intersection[Di, to[x]]]]], fix[composite[to[x], Di]] = True

In[6]:= subclass[range[funpart[intersection[Di, to[x_-]]]]], fix[composite[to[x_-], Di]]] := True

Corollary.

In[7]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]],
  subclass[u, w], {u -> range[funpart[intersection[Di, to[x]]]]},
  v -> fix[composite[to[x], Di]], w -> fix[to[x]]]] // Reverse
Out[7]= subclass[range[funpart[intersection[Di, to[x]]]]], fix[to[x]]] = True

In[8]:= subclass[range[funpart[intersection[Di, to[x_-]]]]], fix[to[x_-]]] := True

Lemma.

In[9]:= SubstTest[subclass, composite[Id, t], cart[y, z], t \[Function] cover[x]] // Reverse
Out[9]= subclass[cover[x], cart[y, z]] =
  and[subclass[domain[cover[x]], y], subclass[range[cover[x]], z]]

In[10]:= subclass[cover[x_-], cart[y_, z_-]] :=
  and[subclass[domain[cover[x]], y], subclass[range[cover[x]], z]]

Theorem.

In[11]:= SubstTest[subclass, range[cover[t]], range[t], t \[Function] to[x]] // Reverse
Out[11]= subclass[range[cover[to[x]]]], fix[to[x]]] = True

In[12]:= subclass[range[cover[to[x_-]]]], fix[to[x_-]]] := True

Lemma. Simplification rule.

In[13]:= equal[composite[id[fix[to[x]]], cover[to[x]]], cover[to[x]]]

In[14]:= composite[id[fix[to[x_-]]], cover[to[x_-]]] := cover[to[x]]

Observation. The immediate successor relation corresponding to the strict part of a total order relation is the cover relation of the total order relation.

In[15]:= immediatesuccessor[intersection[Di, to[x]]]
Out[15]= cover[to[x]]
Corollary. An example.

In[17]:= SubstTest[immediatesuccessor,  
          intersection[Di, to[x]], x -> restrict[S, OMEGA, OMEGA]] // Reverse

Out[17]= composite[id[OMEGA],  
          intersection[E, complement[composite[complement[inverse[E]], id[OMEGA], SUCC, E]]]] :=  
          composite[id[OMEGA], SUCC]

In[18]:= composite[id[OMEGA], intersection[E,  
          complement[composite[complement[inverse[E]], id[OMEGA], SUCC, E]]]] :=  
          composite[id[OMEGA], SUCC]

Corollary. Another example.

In[19]:= SubstTest[immediatesuccessor,  
          intersection[Di, to[x]], x -> restrict[S, omega, omega]] // Reverse

Out[19]= composite[id[omega],  
          intersection[E, complement[composite[complement[inverse[E]], id[omega], SUCC, E]]]] :=  
          composite[id[omega], SUCC]

In[20]:= composite[id[omega], intersection[E,  
          complement[composite[complement[inverse[E]], id[omega], SUCC, E]]]] :=  
          composite[id[omega], SUCC]

**well-order rules**

Since any well-order is a total order relation, the same rewrite rules hold in this special case. Only two additional rewrite rules are needed to deal with this special case.

Corollary.  

In[25]:= SubstTestsubclass,  
          range[funpart[intersection[Di, to[t]]], fix[to[t]], t -> wo[x]] // Reverse


In[26]:= subclass[range[funpart[intersection[Di, wo[x_]]]], fix[wo[x_]]] := True

Corollary

In[28]:= SubstTestcomposite, id[fix[to[t]]], cover[to[t]], t -> wo[x]] // Reverse

Out[28]= composite[id[fix[wo[x]]], cover[wo[x]]] := cover[wo[x]]

In[29]:= composite[id[fix[wo[x_]]], cover[wo[x_]]] := cover[wo[x]]

Observation. The immediate successor relation corresponding to the strict part of a well-order relation is its cover relation.
In[30]:= `immediatesuccessor`[intersection[D_i, wo[x]]]

Out[30]= cover[wo[x]]