one-sided inverses in the category of sets

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summary

This notebook is only concerned with one-sided invertibility in the category of sets. Morphisms in the category of sets are ordered pairs $\text{pair}[u, y]$ such that $u \in \text{map}[x, y]$ where $x = \text{domain}[u]$. When the product of two morphisms in a category is an identity morphism, the left-hand factor is said to be right-invertible or a retraction and the right-hand factor is said to be left-invertible or a section. A membership rule is derived in this notebook for the class $\text{image}[\text{inverse}[\text{CATOFUN}], \text{inverse}[\text{IMAGE}[\text{DUP}]]]$ of ordered pairs $\text{pair}[\text{pair}[u, x], \text{pair}[v, y]]$ of morphisms whose product is an identity morphism. The domain of this class is the class of retractions, and its range is the class of sections. A morphism $\text{pair}[u, x]$ in the category of sets is said to be onto if $x = \text{range}[u]$. A morphism $\text{pair}[v, y]$ in the category of sets is said to be one-to-one if $\text{inverse}[v]$ is a function. It is shown that retractions in the category of sets are onto, and that sections are one-to-one. A counterexample for a converse statement is presented: the one-to-one morphism $\text{PAIR}[0, \text{set}[0]]$ is not a section in the category of sets.

a COMPOSE rule

The composition law $\text{CATOFUN}$ for the category of sets is a certain restriction of the direct product of the two associative functions $\text{COMPOSE}$ and $\text{FIRST}$. A special rewrite rule is needed in this section for the class $\text{inverse}[\text{IMAGE}[\text{DUP}]] \circ \text{COMPOSE}$ of ordered triples $\text{pair}[\text{pair}[x, y], z]$ such that $x \circ y = \text{id}[z]$.

Lemma.

In[2]:= SubstTest[member, pair[w, z], composite[function[funpart[u]], funpart[v]],
            {u \to \text{IMAGE}[\text{DUP}], v \to \text{COMPOSE}, w \to \text{pair}[x, y]]} // Reverse

Out[2]= member[\text{pair}[x, y], z], composite[function[\text{IMAGE}[\text{DUP}]], \text{COMPOSE}] ==
        and[\text{equal}[\text{composite}[x, y], \text{id}[z]], member[x, V], member[y, V], member[z, V]]
Theorem. Membership rule for the one-sided-inverse relation in the category of sets.

Lemma. Reverse implication.

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Theorem. Membership rule for the one-sided-inverse relation in the category of sets.
The following diagram provides a picture for this membership rule.

retractions are onto

Theorem. Retractions in the category of sets are onto.
Theorem. If the product of two morphisms in the category of sets is an inclusion morphism, then the right-hand factor is one-to-one.

Sections in the category of sets are one-to-one. A slightly more general result can be derived.

**Theorem.** If the product of two morphisms in the category of sets is an inclusion morphism, then the right-hand factor is one-to-one.

**Corollary.** Sections in the category of sets are one-to-one.

**Corollary.** A slightly more general result can be derived.
A variable-free formulation of the fact that sections are one-to-one will now be derived.

**Lemma.**

\[ \text{implies member[pair[pair[u, x], pair[v, y]]}, \text{image[Inverse[CATOFUNS], Inverse[IMAGE[DUP]]]}, \text{member[v, BIJ]} \]  

**Theorem.** A variable-free statement that sections in the category of sets are one-to-one.

\[ \text{Map[empty[composite[id[cart[V, V]]}, \text{complement[#]}, \text{id[cart[V, V]]} \] &}, \text{SubstTest[class, pair[pair[u, x], pair[v, y]]}, \text{implies member[pair[pair[u, x], pair[v, y]], s], member[v, t]}], \{s -> \text{image[Inverse[CATOFUNS], Inverse[IMAGE[DUP]]]}, t \rightarrow \text{BIJ}\}] \]

**Corollary.** An upper bound for the class of sections.

\[ \text{Map[empty[composite[id[cart[V, V]]}, \text{complement[#]}, \text{id[cart[V, V]]} \] &}, \text{SubstTest[and, subclass[u, cart[V, V]], subclass[domain[u], v]}, \{u -> \text{range[Inverse[CATOFUNS], Inverse[IMAGE[DUP]]]}, v \rightarrow \text{BIJ}\}] \]

**Corollary.** An upper bound for the one-sided invertibility relation.

\[ \text{Map[empty[composite[id[cart[V, V]]}, \text{complement[#]}, \text{id[cart[V, V]]} \] &}, \text{SubstTest[and, subclass[u, cart[V, V]], subclass[range[u], v]}, \{u -> \text{range[Inverse[CATOFUNS], Inverse[IMAGE[DUP]]]}, v \rightarrow \text{cart[BIJ, V]}\}] \]

This result may be combined with that of the preceding section.
Corollary. An upper bound for the one-sided invertibility relation.

In[34]:= SubstTest[subclass, t, intersection[u, v, w],
{t -> image[inverse[CATOFUNS], inverse[IMAGE[DUP]]], u -> domain[CATOFUNS],
v -> cart[IMAGE[SECOND], V], w -> cart[V, cart[BIJ, V]]}] // Reverse

Out[34]= subclass[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]], cart[
composite[IMAGE[SECOND], id[FUNS]], composite[S, IMAGE[SECOND], id[BIJ]]] ] = True

In[35]:= subclass[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]], cart[
composite[IMAGE[SECOND], id[FUNS]], composite[S, IMAGE[SECOND], id[BIJ]]] ] := True

counterexample

It was shown above that sections in the category of sets are one-to-one. The converse does not hold. In this section a
counterexample is presented that shows that a one-to-one morphism in the category of sets need not be a section.

Lemma. A simplification rule.

In[83]:= equal[composite[Id, image[inverse[image[inverse[CATOFUNS], x]], y]],
image[inverse[image[inverse[CATOFUNS], x]], y]]

Out[83]= True

In[85]:= composite[Id, image[inverse[image[inverse[CATOFUNS], x_]], y_]] :=
image[inverse[image[inverse[CATOFUNS], x]], y]

Lemma. The counterexample.

In[88]:= Map[not[empty[#]] &,
SubstTest[class, pair[u, x], member[pair[pair[u, x], pair[v, y]], t],
{t -> image[inverse[CATOFUNS], inverse[IMAGE[DUP]]], u -> 0, v -> set[0]}]]

Out[88]= member[pair[0, set[0]], range[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]]]] ] = False

In[89]:= member[pair[0, set[0]], range[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]]]] ] := False

Theorem. A one-to-one morphism in the category of sets need not be a section.

In[91]:= Map[not, SubstTest[implies, and[member[x, s], subclass[s, t]]],
member[x, t], {x -> pair[0, set[0]], s -> composite[S, IMAGE[SECOND], id[BIJ]],
t -> range[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]]]}] // Reverse

Out[91]= subclass[composite[S, IMAGE[SECOND], id[BIJ]],
range[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]]]] ] = False

In[92]:= subclass[composite[S, IMAGE[SECOND], id[BIJ]],
range[image[inverse[CATOFUNS], inverse[IMAGE[DUP]]]]] ] := False