inverses of powers of group elements

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summary

The composite of a functor from one monoid to another with the power list of an element of the range of a monoid is the list of powers of a transformed element of the range of the second monoid. Since \texttt{inv[gp[x]]} is a functor from \texttt{gp[x]} to \texttt{gp[x] \& SWAP}, one can show that powers of inverses are inverses of powers.

functors for groups

For groups, functors are the same as binary homomorphisms.

Lemma.

\begin{verbatim}
In[2]:= SubstTest[implies, and[member[u, GROUPS], member[v, GROUPS], member[t, binhom[u, v]]],
functor[t, u, v], {u \rightarrow gp[x], v \rightarrow gp[y]}] // Reverse
Out[2]= or[equal[0, gp[x]], equal[0, gp[y]],
functor[t, gp[x], gp[y]], not[member[t, binhom[gp[x], gp[y]]]]] = True

In[3]:= (\% / {t \rightarrow t\_, x \rightarrow x\_, y \rightarrow y\_}) /. Equal \rightarrow SetDelayed

Lemma. (Eliminate redundant literals.)
\end{verbatim}
Corollary. In the special case of \texttt{inv[gp][x]}, one has this corollary.
Reverse

Lemma.

One can replace right multiplication with left multiplication in the present situation.

Lemma.

There is a redundant literal.

Lemma.

It is not clear how best to orient the following. The choice was made to eliminate APPLY.

Main Theorem. Inverses of powers are powers of inverses for group elements.
In[19]:= iterate[composite[gp[x_], LEFT[APPLY[inv[gp[x_]], y_]]], set[e[gp[x_]]]] := composite[inv[gp[x]], iterate[composite[gp[x], LEFT[y]], set[e[gp[x]]]]]

Corollary.

In[20]:= SubstTest[iterate, composite[gp[u], LEFT[APPLY[inv[gp[u]], v]]], set[e[gp[u]]], {u \rightarrow \text{INTADD}, v \rightarrow \text{int}[x]}] // Reverse

Out[20]= iterate[inverse[intplus[int[x]]], set[id[\text{omega}]]]
    := composite[\text{INVERSE, MIXMUL, LEFT}[\text{int}[x]]]

In[21]:= iterate[inverse[intplus[int[x_]]], set[id[\text{omega}]]] := composite[\text{INVERSE, MIXMUL, LEFT}[\text{int}[x]]]

The orientation of the following was suggested by examining the special case that \text{int}[x] is \text{plus}[x].

Theorem.

In[22]:= SubstTest[iterate, intplus[int[t]], set[id[\text{omega}]], t \rightarrow \text{inverse}[\text{int}[x]]]

Out[22]= \text{composite}[\text{MIXMUL, LEFT}[\text{inverse}[\text{int}[x]]]] = \text{composite}[\text{INVERSE, MIXMUL, LEFT}[\text{int}[x]]]

A more general result can be derived.

Theorem.

In[23]:= Assoc[\text{MIXMUL, cross}[\text{INVERSE, Id}], LEFT[\text{composite}[\text{Id, x}]]]

Out[23]= \text{composite}[\text{MIXMUL, LEFT}[\text{inverse}[\text{int}[x]]]] = \text{composite}[\text{INVERSE, MIXMUL, LEFT}[\text{composite}[\text{Id, x}]]]

In[24]:= composite[\text{MIXMUL, LEFT}[\text{inverse}[\text{x_}]]] :=
    \text{composite}[\text{INVERSE, MIXMUL, LEFT}[\text{composite}[\text{Id, x}]]]