isomorphism is an equivalence relation

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An isomorphism in an (arrows-only) category cat\[x\] is an invertible morphism, that is, a morphism \(u\) for which there exists another morphism \(v\) such that both \(u \cdot v\) and \(v \cdot u\) are identity morphisms. The class of isomorphisms \(\text{domain[inv[cat[x]]]}\) is a subclass of the class \(\text{range[cat[x]]}\) of all morphisms. In an arrows-only category \(\text{cat[x]}\), the role of objects is played by identity morphisms. The class \(\text{ids[cat[x]]}\) of all identity morphisms is a subclass of the class of isomorphisms.

Recall that the relation \(\text{domain[hom[cat[x]]]}\) consists of all pairs of identities \(u, v\) for which there is a morphism from \(u\) to \(v\). This transitive relation is a subclass of the cartesian square of the class \(\text{ids[cat[x]]}\). The focus in this notebook is a certain subrelation of the relation \(\text{domain[hom[cat[x]]]}.\)

Two identity morphisms \(u\) and \(v\) are isomorphic if there exists an invertible morphism from \(u\) to \(v\). This isomorphism relation on the class \(\text{ids[cat[x]]}\) is given by the expression \(\text{image[inverse[hom[cat[x]]], domain[inv[cat[x]]]}}\). It will be shown below that this isomorphism relation on identity morphisms is an equivalence relation. An interesting feature of the derivation is that it can be accomplished entirely without introducing variables for morphisms.

The derivation is:

The class of \(\text{domain[inv[cat[x]]}\) of isomorphisms is binary-closed under the composition law \(\text{cat[x]}\). This fact can be strengthened to an equation and made into a rewrite rule:

Claim. The class of products of isomorphisms is equal to the class of isomorphisms.
domain[domain[cat[x]]] is transitive

Lemma.

Map[composite[#, cross[DUP, DUP]] &,
    (composite[SWAP, RIF, cross[cross[u, v], cross[u, v]] // ReifTriNormality] /.,
     {u -> dom[cat[x]], v -> cod[cat[x]]})

In[5]:= Map[composite[#, cross[DUP, DUP]] &,
    (composite[SWAP, RIF, cross[cross[u, v], cross[u, v]] // ReifTriNormality] /.,
     {u -> dom[cat[x]], v -> cod[cat[x]]})

    composite[SWAP, cross[dom[cat[x]], cod[cat[x]]], id[dom[cat[x]]]]

In[6]:= composite[SWAP, RIF, cross[inverse[dom[cat[x]]], inverse[dom[cat[x]]]]] =
    composite[SWAP, cross[dom[cat[x]], cod[cat[x]]], id[dom[cat[x]]]]
The \textbf{cod} of the inverse of an invertible morphism is the \textbf{dom} of the morphism.

Lemma.

An application of duality provides the following corollary.

Corollary. Equation for the \textbf{dom} of an inverse.
Theorem. The function \texttt{inv\[cat\[x\]]} is an involution.

\texttt{In[27]:=} \texttt{SubstTest[composite, funpart[t], inverse[funpart[t]], t \rightarrow inv[cat[x]]]} // Reverse
\texttt{Out[27]=} \texttt{composite[inv[cat[x]], inv[cat[x]]] = id[domain[inv[cat[x]]]]}

\texttt{In[28]:=} \texttt{composite[inv[cat[x]], inv[cat[x]]] := id[domain[inv[cat[x]]]]}

Theorem. An equation for the composite of \texttt{inv[cat[x]]} and \texttt{hom[cat[x]]}.

\texttt{In[30]:=} \texttt{Map[composite[inv[cat[x]], inverse[#], SWAP \&, Assoc[}
\texttt{cross[dom[cat[x]], cod[cat[x]]], cross[inv[cat[x]], inv[cat[x]]], DUP]} // Reverse
\texttt{Out[30]=} \texttt{composite[inv[cat[x]], hom[cat[x]]] =}
\texttt{composite[ids[domain[inv[cat[x]]]], hom[cat[x]], SWAP]}

\texttt{In[31]:=} \texttt{composite[inv[cat[x]], hom[cat[x]]] :=}
\texttt{composite[ids[domain[inv[cat[x]]]], hom[cat[x]], SWAP]}

Theorem. Isomorphism is a symmetric relation.

\texttt{In[32]:=} \texttt{IminComp[inv[cat[x]], hom[cat[x]], V]}
\texttt{Out[32]=} \texttt{inverse[domain[inv[cat[x]]], domain[inv[cat[x]]]] =}
\texttt{image[inv[cat[x]], domain[inv[cat[x]]]]}

\texttt{In[33]:=} \texttt{inverse[domain[inv[cat[x]]], domain[inv[cat[x]]]] :=}
\texttt{image[inv[cat[x]], domain[inv[cat[x]]]]}

Corollary. Isomorphism is an equivalence relation.

\texttt{In[34]:=} \texttt{SubstTest[and, SYMMETRIC[t], TRANSITIVE[t],}
\texttt{t \rightarrow image[inv[hom[cat[x]]], domain[inv[cat[x]]]]]}
\texttt{Out[34]=} \texttt{EQUIVALENCE[image[inv[hom[cat[x]]], domain[inv[cat[x]]]]] = True}

\texttt{In[35]:=} \texttt{EQUIVALENCE[image[inv[hom[cat[x]]], domain[inv[cat[x]]]]] := True}

\underline{fix point class}

Lemma.

\texttt{In[38]:=} \texttt{SubstTest[implies, subclass[u, v], subclass[fix[u], fix[v]],}
\texttt{u \rightarrow image[inv[hom[cat[x]]], domain[inv[cat[x]]]],}
\texttt{v \rightarrow cart[ids[cat[x]], ids[cat[x]]]} // Reverse
\texttt{Out[38]=} \texttt{subclass[fix[image[inv[hom[cat[x]]], domain[inv[cat[x]]]]], ids[cat[x]]] = True}

\texttt{In[39]:=} \texttt{(% / . x \rightarrow x\_)} / . \texttt{Equal} \rightarrow \texttt{SetDelayed}

Lemma.
Theorem. Every identity morphism is isomorphic to itself.

In[45]:=
SubstTest[and, subclass[u, v], subclass[v, u],  
{u -> fix[image[inverse[hom[cat[x]]]], domain[inv[cat[x]]]]}, v -> ids[cat[x]]]  

Out[45]=  
Equal[fix[image[inverse[hom[cat[x]]]], domain[inv[cat[x]]]], ids[cat[x]]] = True  

In[48]:=
fix[image[inverse[hom[cat[x]]]], domain[inv[cat[x]]]] := ids[cat[x]]