iterate[K, singleton[0]]

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In[1]:= << goede52.t18; << tools.m;

: Package Title: goede52.t18 2003 November 28 at 11:15 p.m.
It is now: 2003 Dec 9 at 10:29
Loading Simplification Rules
TOOLS.M Revised 2003 November 15
weightlimit = 40

summary

It is shown in this notebook that adding a new element to a finite set increases the number of elements by one. The relation iterate[K, singleton[0]] is shown to have as its $n$-th vertical section the class of all sets with cardinality $n$ for every natural number $n$. Some additional formulas relating the cardinality function $\text{CARD}$, the equipollence relation $\mathcal{Q}$ and the cover relation $K = \text{dif}[\mathcal{PS}, \text{composite}[\mathcal{PS}, \mathcal{PS}]]$ for finite sets are derived. Here $\mathcal{PS} = \text{dif}[\mathcal{S}, \text{Id}]$ is the proper subset relation.

a function lemma

The lemma in this section is used in the next section to derive a formula for the restriction of $K$ to $\omega$.

In[2]:= SubstTest[implies, subclass[u, v], subclass[composite[x, u, y], composite[x, v, y]],
{u -> composite[K, inverse[K]], v -> Q, x -> id[omega], y -> id[omega]}]
Out[2]= subclass[composite[id[omega], K, inverse[K], id[omega]], id[omega]] := True

In[3]:= subclass[composite[id[omega], K, inverse[K], id[omega]], id[omega]] := True

In[4]:= SubstTest[subclass, composite[x, inverse[x]],
id[omega], x -> composite[id[omega], K]] // Reverse
Out[4]= FUNCTION[composite[id[omega], K]] := True

In[5]:= FUNCTION[composite[id[omega], K]] := True

connecting $K$ and SUCC

The first lemma follows from the fact that natural numbers do not belong to themselves.
A connection between $\mathbf{K}$ and $\mathbf{CUP}$ has been explored previously, but adding an extra factor of $\text{id}[\mathbf{DISJOINT}]$ yields a somewhat cleaner result:

In[8]:  
composite[CUP, id[DISJOINT],  
  id[cart[V, range[SINGLETON]]], inverse[\mathbf{FIRST}]] // RelnRenormality

Out[8]=  
composite[CUP, id[composite[id[range[SINGLETON]], \mathbf{DISJOINT}]], inverse[\mathbf{FIRST}]] = K

In[9]:  
composite[CUP, id[composite[id[range[SINGLETON]], \mathbf{DISJOINT}]], inverse[\mathbf{FIRST}]] := K

From this one derives an inclusion:

In[10]:  
SubstTest[implies, subclass[u, v], subclass[composite[x, u, y], composite[x, v, y]],  
  \{u -> id[composite[SINGLETON, id[omega]]],  
  v -> id[composite[id[range[SINGLETON]], \mathbf{DISJOINT}]], x -> CUP, y -> inverse[\mathbf{FIRST}]\}]

Out[10]=  
subclass[composite[id[omega], SUCC], K] = True

In[11]:  
subclass[composite[id[omega], SUCC], K] := True

As a corollary, one obtains a formula for the restriction of the cover relation $\mathbf{K}$ to the set $\omega$ of natural numbers.

In[12]:  
SubstTest[implies, and[subclass[x, y], FUNCTION[y]],  
  equal[x, composite[y, id[domain[x]]]],  
  \{x -> composite[id[omega], SUCC], y -> composite[id[omega], K]\}]

Out[12]=  
equal[composite[id[omega], SUCC], composite[id[omega], K, id[omega]]] = True

In[13]:  
composite[id[omega], K, id[omega]] := composite[id[omega], SUCC]

The inverse formula will also be used below.

In[14]:  
composite[id[omega], inverse[K], id[omega]] // DoubleInverse

Out[14]=  
composite[id[omega], inverse[K], id[omega]] = composite[inverse[SUCC], id[omega]]

In[15]:  
composite[id[omega], inverse[K], id[omega]] := composite[inverse[SUCC], id[omega]]

another function lemma

To derive the main theorem in this notebook, another function lemma is used:

In[16]:  
SubstTest[implies, subclass[x, y], subclass[composite[u, x, v], composite[u, y, v]],  
  \{u -> composite[id[omega], \mathbf{CARD}], v -> composite[inverse[\mathbf{CARD}], id[omega]],  
  x -> composite[K, inverse[K]], y -> Q\}]

Out[16]=  
subclass[  
  composite[id[omega], \mathbf{CARD}, K, inverse[K], inverse[\mathbf{CARD}], id[omega]], id[omega]] = True

In[17]:  
subclass[composite[id[omega], \mathbf{CARD}, K, inverse[K], inverse[\mathbf{CARD}], id[omega]], id[omega]] := True
In[18]:= SubstTest[subclass, composite[x, inverse[x]], id[omega],
    x -> composite[id[omega], CARD, inverse[K]]] // Reverse
Out[18]= FUNCTION[composite[id[omega], CARD, inverse[K]]] = True

In[19]:= FUNCTION[composite[id[omega], CARD, inverse[K]]] := True

In[20]:= SubstTest[implies, subclass[u, v], subclass[composite[u, w], composite[v, w]],
    {u -> restrict[inverse[K], omega, omega], v -> inverse[K], w -> CARD}]
Out[20]= subclass[composite[composite[inverse[SUCC], id[omega], CARD],
    composite[inverse[K], CARD]], composite[CARd, inverse[K]]] = True

In[21]:= subclass[composite[inverse[SUCC], id[omega], CARD],
    composite[inverse[K], CARD]] := True

In[22]:= SubstTest[implies, subclass[u, v], subclass[composite[w, u], composite[w, v]],
    {u -> CARD, v -> Q, w -> composite[id[omega], inverse[K]]}]
Out[22]= subclass[composite[id[omega], inverse[K], CARD],
    composite[CARd, inverse[K]]] = True

In[23]:= subclass[composite[id[omega], inverse[K], CARD],
    composite[CARd, inverse[K]]] := True

In[24]:= SubstTest[implies, and[subclass[u, v], subclass[v, w]], subclass[u, w],
    {u -> composite[composite[inverse[SUCC], id[omega], CARD],
        v -> composite[id[omega], inverse[K], CARD],
        w -> composite[id[omega], CARD, inverse[K]]}]
Out[24]= subclass[composite[composite[inverse[SUCC], id[omega], CARD],
    composite[CARd, inverse[K]]] := True

In[25]:= subclass[composite[inverse[SUCC], id[omega], CARD],
    composite[CARd, inverse[K]]] := True

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CARD, Q and FINITE

In this section further simplification rules involving the cardinality function CARD, the equipollence relation Q and the class FINITE of finite sets are derived.

In[26]:= equal[composite[CARD, id[FUTURE]], composite[id[omega], CARD]] // AssertTest
Out[26]= equal[composite[CARD, id[FUTURE]], composite[id[omega], CARD]] = True

In[27]:= composite[CARD, id[FUTURE]] := composite[id[omega], CARD]

Lemma. The class of finite sets is contained in the domain of the function CARD.

In[28]:= equal[intersection[FUTURE, image[Q, OMEGA]], FUTURE]
Out[28]= True

In[29]:= intersection[FUTURE, image[Q, OMEGA]] := FUTURE

Equipollence for finite sets is equivalent to equality of their cardinalities:

In[30]:= Assoc[Q, id[image[Q, OMEGA]], id[FUTURE]]
Out[30]= composite[Q, id[FUTURE]] = composite[inverse[CARD], id[omega], CARD]
Lemma. The cardinality of a set is a nonzero natural number if and only if the set is finite and nonempty.

The final step is to use the uniqueness theorem for *iterate*.

This formula implies that for each natural number *n*, the class of sets with cardinality *succ[n]* is the image under *K* of the class of sets with cardinality *n*.
In this section, a formula is derived for another function obtained by replacing `inverse[K]` with `K`.

An explicit formula for the function `composite[id[omega], CARD, K]` follows:

An explicit formula for the function `composite[id[omega], CARD, K]` follows:

**Corollary.** The cover relation commutes with the restriction of the equipollence relation to finite sets.