the category of sets is locally small

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An (arrows-only) category \(\text{cat}[x]\) is said to be \textbf{locally small} if for every pair of identity morphisms \(u, v \in \text{ids}[\text{cat}[x]]\), the class of morphisms from \(u\) to \(v\) is a set. It is convenient to reformulate this condition in a fashion that avoids the explicit quantification over the set variables \(u\) and \(v\). The key to hiding these quantifiers is to make use of the ternary relation \(\text{hom}[\text{cat}[x]]\) which by definition consists of all ordered triples \(\text{pair}[\text{pair}[u, v], w]\) such that \(w\) is a morphism from \(u\) to \(v\). In other words, the membership statement \(\text{pair}[\text{pair}[u, v], w] \in \text{hom}[\text{cat}[x]]\) holds if and only if the morphism \(w \in \text{range}[\text{cat}[x]]\) satisfies the equations \(u = \text{APPLY}[	ext{dom}[\text{cat}[x]], w]\) and \(v = \text{APPLY}[	ext{cod}[\text{cat}[x]], w]\). The assertion that a category \(\text{cat}[x]\) is locally small therefore translates into the condition that every vertical section \(\text{image}[\text{hom}[\text{cat}[x]], \text{set}[\text{PAIR}[u, v]]]\) of the ternary relation \(\text{hom}[\text{cat}[x]]\) is a set. Equivalently, a category \(\text{cat}[x]\) is locally small if and only if the corresponding ternary relation \(\text{hom}[\text{cat}[x]]\) is thin, which can be stated concisely as the equation \(\text{domain}[- \text{VERTSECT}[\text{hom}[\text{cat}[x]]]] = \text{V}\). In this notebook it is shown that the category of sets \(\text{CATOFUNS}\) is locally small.

**derivation**

In the category of sets each identity morphism is of the form \(\text{PAIR}[\text{id}[u], u]\), where \(u\) is a set. Each vertical section of the ternary relation \(\text{hom}[\text{CATOFUNS}]\) can be expressed as follows in terms of the set \(\text{map}[u, v]\) of mappings from \(u\) to \(v\).

**Theorem.** An explicit formula for a typical vertical section of the ternary relation \(\text{hom}[\text{CATOFUNS}]\).
This can be cleaned up by making the use of the function which takes each set to the corresponding identity morphism:

\[
\text{In}[4] := \text{VERTSECT[reify[u, PAIR[id[u], u]]]}
\]

\[
\text{Out}[4] = \text{composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]]}
\]

Lemma.  (Eliminating the variables \(u\) and \(v\) from the statement that the vertical section at any pair of identity morphisms is a set.)

\[
\text{In}[5] := \text{Map[empty[composite[Id, complement[#]]] &,
      SubstTest[class, pair[u, v], member[image[t, set[PAIR[u, v]]], V], t ->
      composite[hom[CATOFUNS], cross[composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]],
      composite[id[inverse[IMAGE[DUP]]], inverse[SECOND]]]]]}
\]

\[
\text{Out}[5] = \text{subclass[cart[inverse[IMAGE[DUP]]], inverse[IMAGE[DUP]]],
  domain[VERTSECT[hom[CATOFUNS]]] = True}
\]

In[6]:= % /. Equal \[\rightarrow\] SetDelayed

Theorem.  The category of sets is locally small.

\[
\text{In}[7] := \text{SubstTest[implies, and[subclass[u, v], subclass[v, w]],
  subclass[u, w], \{u \rightarrow \text{domain[hom[CATOFUNS]], v \rightarrow \text{cartsq[ids[CATOFUNS]]},
  w \rightarrow \text{domain[VERTSECT[hom[CATOFUNS]]]]\} // Reverse}
\]

\[
\text{Out}[7] = \text{equal[V, domain[VERTSECT[hom[CATOFUNS]]]] = True}
\]

\[
\text{In}[8] := \text{domain[VERTSECT[hom[CATOFUNS]]] := V}
\]