NATEXP, part 4.

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In [1] := SetDirectory["1:"]; << goedel83.01a; << tools.m

:Package Title: goedel83.01a 2006 July 1 at 11:50 a.m.

It is now: 2006 Jul 2 at 7:13

Loading Simplification Rules

TOOLS.M Revised 2006 June 27

weightlimit = 40

summary

In this fourth notebook in a series about exponentiation for natural numbers it is shown by induction that the number of subsets of a natural number \( n \) is \( 2^n \). More generally, the cardinality of the power set of any finite set \( x \) is equal to \( 2^{\text{card}(x)} \).

ADJOIN

The restriction of \( \text{ADJOIN}[\text{set}[x]] \) to the power set of a natural number \( x \) is one-to-one.


This function is also finite.

In [4] := SubstTest[member, domain[funpart[w]], FINITE, w \[Rule\] composite[ADJOIN[set[nat[x]]]], id[P[nat[x]]]]] // Reverse


The domain and range of a finite bijection have equal cardinality.
The union of the power set of a natural number \( x \) and its image under \( \text{ADJOIN}[\text{set}[x]] \) is the power set of \( \text{succ}[x] \).

The image of \( x \) under \( \text{ADJOIN}[\text{set}[x]] \) is finite.

The cardinality of the union of two disjoint finite sets is the sum of their cardinalities.

**base case**

Lemma.

The image, \( \text{union} \), complement, \( \text{cart} \), \( \text{range} \) and \( \text{id} \) of a finite set is finite.

The union of the power set of a natural number \( x \) and its image under \( \text{ADJOIN}[\text{set}[x]] \) is the power set of \( \text{succ}[x] \).

The image of \( x \) under \( \text{ADJOIN}[\text{set}[x]] \) is finite.

The cardinality of the union of two disjoint finite sets is the sum of their cardinalities.
The 0th power of any number is 1. For a non-number, the 0th power is V.

\[ \text{In [16]} := \text{Map[A, ImageComp[NATEXP, RIGHT[0], set[x]]]} \text{ // Reverse} \]
\[ \text{Out[16]} = \text{natexp[x, 0]} = \text{union[complement[image[V, intersection[omega, set[x]]]], set[0]]} \]
\[ \text{In[17]} := \text{natexp[x_, 0]} := \text{union[complement[image[V, intersection[omega, set[x]]]], set[0]]} \]

These results suffice for the base case.

\[ \text{In [18]} := \text{equal[card[P[0]], natexp[succ[set[0]], 0]]} \]
\[ \text{Out[18]} = \text{True} \]

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**induction step**

In general, one obtains the next power of a natural number x by multiplying by another factor of x.

\[ \text{In [19]} := \text{SubstTest[natexp, nat[x], natadd[nat[y], z], z \rightarrow set[0]]} \]
\[ \text{Out[19]} = \text{natexp[nat[x], succ[nat[y]]]} = \text{natmul[nat[x], natexp[nat[x], nat[y]]]} \]
\[ \text{In [20]} := \text{natexp[nat[x_], succ[nat[y_]]]} := \text{natmul[nat[x], natexp[nat[x], nat[y]]]} \]

For the case \( x = 2 \), multiplying a number by 2 is equivalent to adding the number to itself.

\[ \text{In [21]} := \text{SubstTest[natexp, nat[w], succ[nat[x]], w \rightarrow succ[set[0]]]} \]
\[ \text{Out[21]} = \text{natexp[succ[set[0]], succ[nat[x]]]} = \text{natmul[natexp[succ[set[0]], nat[x]], natexp[succ[set[0]], nat[x]]]} \]
\[ \text{In [22]} := \text{natexp[succ[set[0]], succ[nat[x_]]]} := \text{natmul[natexp[succ[set[0]], nat[x]], natexp[succ[set[0]], nat[x]]]} \]

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**induction**

To carry out an induction argument using the GOEDEL program, one needs to construct an inductive class. To construct this class, one can use the fact that the combination of VERTSECT with reify is equivalent to lambda.

\[ \text{In [23]} := \text{VERTSECT[reify[x, card[P[x]]]]} \]
\[ \text{Out[23]} = \text{composite[CARD, POWER]} \]
\[ \text{In [24]} := \text{VERTSECT[reify[x, natexp[succ[set[0]], x]]]} \]
\[ \text{Out[24]} = \text{composite[NATEXP, LEFT[succ[set[0]]]]} \]

One is thus led to consider the following class.
The following membership rule is needed to work with this class.

One also needs a special membership rule for NATEXP.

Finally, one needs to use the fact that any finite set belongs to the class image[Q, OMEGA].

The induction step needs to be rewritten without nat wrappers so that one can eliminate the variables.
At this point, one can eliminate the variable $x$.

Applying induction, one finds:

Finally, variables are reintroduced to obtain the main theorem:

\[
\text{Lemma.}
\]

\[
\text{variable-free formulation}
\]
Variable-free formulation of the theorem derived in the preceding section.

\[ \text{Map} \{ \text{composite}[\text{VERTSECT}[^1], \text{id}[\text{omega}]] & , \\
\text{SubstTest}[\text{reify}, x, \text{card}[P[f[x]]], f \rightarrow \text{nat}] \} /\text{Reverse} \]

\[ \text{composite}[\text{CARD}, \text{POWER}, \text{id}[\text{omega}]] = \text{composite}[\text{NATEXP}, \text{LEFT}[\text{succ}[\text{set}[0]]] \]

\[ \text{composite}[\text{CARD}, \text{POWER}, \text{id}[\text{omega}]] := \text{composite}[\text{NATEXP}, \text{LEFT}[\text{succ}[\text{set}[0]]] \]

**generalization to arbitrary finite sets**

Lemma.

\[ \text{Map} \{ \text{member}[\text{fin}[y], ^1] & , \text{ImageComp}[Q, \text{id}[\text{FINITE}], \text{set}[\text{nat}[x]]] \} /\text{Reverse} \]

\[ \text{member}[\text{pair}[\text{nat}[x], \text{fin}[y]], Q] = \text{equal}[\text{card}[\text{fin}[y]], \text{nat}[x]] \]

\[ \text{member}[\text{pair}[\text{nat}[x_] , \text{fin}[y_]], Q] := \text{equal}[\text{card}[\text{fin}[y]], \text{nat}[x]] \]

More generally, equipollence of finite sets is equivalent to the statement that their cardinalities are equal.

\[ \text{Map} \{ \text{member}[\text{fin}[y], ^1] & , \text{ImageComp}[Q, \text{id}[\text{FINITE}], \text{set}[\text{fin}[x]]] \} /\text{Reverse} \]

\[ \text{member}[\text{pair}[\text{fin}[x], \text{fin}[y]], Q] = \text{equal}[\text{card}[\text{fin}[x]], \text{card}[\text{fin}[y]]] \]

\[ \text{member}[\text{pair}[\text{fin}[x_], \text{fin}[y_]], Q] := \text{equal}[\text{card}[\text{fin}[x]], \text{card}[\text{fin}[y]]] \]

Lemma.

\[ \text{SubstTest}[\text{member}, \text{pair}[\text{fin}[u], \text{fin}[v]], \\
Q, \{ u \rightarrow \text{P}[\text{nat}[x]], v \rightarrow \text{P}[\text{fin}[y]] \}] /\text{.} x \rightarrow \text{card}[\text{fin}[y]] \]

\[ \text{member}[\text{pair}[\text{P}[\text{card}[\text{fin}[y]]], \text{P}[\text{fin}[y]]], Q] = \\
\text{equal}[\text{card}[\text{P}[\text{fin}[y]]], \text{natexp}[\text{succ}[\text{set}[0]], \text{card}[\text{fin}[y]]] \]

\[ \text{member}[\text{pair}[\text{P}[\text{card}[\text{fin}[y_]]], \text{P}[\text{fin}[y_]]], Q] := \\
\text{equal}[\text{card}[\text{P}[\text{fin}[y]]], \text{natexp}[\text{succ}[\text{set}[0]], \text{card}[\text{fin}[y]]] \]

Equipollent sets have equipollent power sets. It follows that the cardinality of the power set of a finite set is equal to 2 raised to the power of the cardinality of the set.
In[54]:= \text{SubstTest}[\text{implies, and}[\text{member}[u, v], \text{subclass}[v, w]], \text{member}[u, w],
\{u \rightarrow \text{pair}[\text{nat}[x], \text{fin}[y]], v \rightarrow Q, w \rightarrow \text{composite}[\text{inverse}[\text{POWER}], Q, \text{POWER}]\}] /.
x \rightarrow \text{card}[\text{fin}[y]]

Out[54]= \text{equal}[\text{card}[P[\text{fin}[y]]], \text{natexp}[\text{succ}[\text{set}[0]], \text{card}[\text{fin}[y]]]] = \text{True}

In[55]:= \text{card}[P[\text{fin}[x_]]] := \text{natexp}[\text{succ}[\text{set}[0]], \text{card}[\text{fin}[x]]]

A variable-free restatement of this general result:

In[56]:= \text{Map}[\text{composite}[\text{VERTSECT}[\#], \text{id}[\text{FINITE}]] \&,
\text{SubstTest}[\text{reify}, x, \text{card}[P[f[x]]], f \rightarrow \text{fin}]] \text{// Reverse}

Out[56]= \text{composite}[\text{CARD, POWER, id}[\text{FINITE}]] = \text{composite}[\text{NATEXP, LEFT}[\text{succ}[\text{set}[0]]], \text{CARD}]

In[57]:= \text{composite}[\text{CARD, POWER, id}[\text{FINITE}]] := \text{composite}[\text{NATEXP, LEFT}[\text{succ}[\text{set}[0]]], \text{CARD}]