Theorem ON-IND-4

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**summary**

Theorem **ON-IND-4**, proved 2000 February 1 using McCune's automated reasoning program **Otter**, is a version of ordinary induction that does not specify any particular base case. The theorem amounts to the statement that any nonempty subset $x$ of the set $\omega$ of natural numbers which is successor invariant, is the relative complement in $\omega$ of its least member. The result is formulated as an equation involving $A[x]$ that remains true when the set is empty; in that case $A[x]$ is not the least member of $x$, but instead is equal to $V$. The **Otter** proof used a lemma stating that $x$ and $A[x]$ are disjoint for any class of ordinal numbers. In fact, this is the case for any subclass of **RUSSELL**. This fact is also used in the derivation presented in this notebook. The disjointness of $x$ and $A[x]$ is rederived below for the special case of natural numbers.

**lemmas**

Lemma 1. Any natural number is the least member of its relative complement in $\omega$.

$$\begin{align*}
\text{In}[2]: & = \text{SubstTest}[[\text{implies}, \text{and}[\text{equal}[\text{dif}[\omega, y], x], \text{member}[y, \omega]], \\
& \quad \text{equal}[A[x], y], y \rightarrow \text{dif}[\omega, x]] \\
\text{Out}[2]: & = \text{or}[\text{equal}[A[x], \text{intersection}[\omega, \text{complement}[x]]], \\
& \quad \text{not}[\text{member}[\text{intersection}[\omega, \text{complement}[x]], \omega]], \\
& \quad \text{not}[\text{subclass}[x, \omega]]] = \text{True}
\end{align*}$$
Lemma 2. The relative complement in \( \text{omega} \) of any non-empty successor-invariant set of natural numbers is a natural number.

The combination of the above two lemmas yields an equation involving \( A[x] \), but it is not quite the result that is wanted.

Lemma.

x and \( A[x] \) are disjoint for any set of natural numbers

Theorem.
Theorem ON-IND-4

The hypothesis that \( x \) is not empty can be eliminated by transposing some terms in the equation in the conclusion of this lemma.

\[
In[12]:= \text{Map[not, SubstTest[and, implies[and[p1, p2], or[p3, p4]],}
\text{implies[p3, p6], implies[p1, p5], implies[and[p1, p4, p5], p6],}
\text{not[implies[and[p1, p2], p6]], \{p1 \rightarrow \text{subclass}[x, \text{omega}],}
\text{p2 \rightarrow \text{invariant[SUCC, x], p3 \rightarrow equal[0, x], p4 \rightarrow equal[A[x], \text{dif[omega, x]]},}
\text{p5 \rightarrow \text{disjoint}[x, A[x]], p6 \rightarrow equal[x, \text{dif[omega, A[x]]}]\}]}\]
\]

\[
Out[12]= \text{or[equal[x, intersection[omega, complement[A[x]]]],}
\text{not[subclass[x, \text{omega}]], not[subclass[\text{image[SUCC, x]}, x]]] = True}
\]

\[
In[13]:= \text{or[equal[x_, intersection[omega, complement[A[x_]]]],}
\text{not[subclass[x_, \text{omega}]], not[subclass[\text{image[SUCC, x_]}, x_]]] = True}
\]