x has no infinite subsets iff P[x] does not

Johan G. F. Belinfante
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In[1]: = SetDirectory["1: "; << goedel79.01a; << tools.m
:Package Title: goedel79.01a 2006 March 1 at 6:30 p.m.
It is now: 2006 Mar 2 at 14:54
Loading Simplification Rules
TOOLS.M Revised 2006 February 3
weightlimit = 40

summary

A class has no infinite subsets if and only if its power class has no infinite subsets.

derivation

If y is an infinite subset of x, then P[y] is an infinite subset of P[x].

In[2]: = SubstTest[implies, subclass[u, v], subclass[image[w, u], image[w, v]],
   {u -> P[P[x]], v -> FINITE, w -> inverse[POWER]}]
Out[2]= or[not{subclass[P[P[x]], FINITE]}, subclass[P[x], FINITE]] = True

In[3]: = (% /. x -> x_)/. Equal -> SetDelayed

The converse also holds:

In[4]: = SubstTest[implies, subclass[u, v],
   subclass[image[w, u], image[w, v]], {u -> P[x], v -> FINITE, w -> inverse[BIGCUP]}]
Out[4]= or[not{subclass[P[x], FINITE]}, subclass[P[x], FINITE]] = True

In[5]: = (% /. x -> x_)/. Equal -> SetDelayed

Combining these two results yields an interesting rewrite rule:

In[6]: = equiv[subclass[P[P[x]], FINITE], subclass[P[x], FINITE]]
In[7]:= \text{subclass}[\text{P}[x_\text{\_}], \text{FINITE}] := \text{subclass}[\text{P}[x], \text{FINITE}]

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the case of a set

When \( x \) is a set, the condition \( \text{subclass}[\text{P}[x], \text{FINITE}] \) is equivalent to the statement that \( x \) is finite.

\text{In[8]}:= \text{SubstTest}[\text{implies},\text{and}[[\text{member}[y, V], \text{subclass}[\text{P}[y], \text{FINITE}]],\text{member}[y, \text{FINITE}], y \to \text{setpart}[x]]

\text{Out[8]}= \text{or}[\text{member}[\text{setpart}[x], \text{FINITE}], \text{not}[\text{subclass}[\text{P}[\text{setpart}[x]], \text{FINITE}]]] = \text{True}

\text{In[9]}:= (\% / . x_\text{\_} / . \text{Equal} \to \text{SetDelayed})

\text{In[10]}= \text{equiv}[\text{subclass}[\text{P}[\text{setpart}[x]], \text{FINITE}], \text{member}[\text{setpart}[x], \text{FINITE}]]

\text{Out[10]}= \text{True}

\text{In[11]}= \text{subclass}[\text{P}[\text{setpart}[x_\text{\_}]], \text{FINITE}] := \text{member}[\text{setpart}[x], \text{FINITE}]

On account of this, the rewrite rule derived in the preceding section could be deduced for the case of sets from this fact:

\text{In[12]}= \text{member}[\text{P}[x], \text{FINITE}]

\text{Out[12]}= \text{member}[x, \text{FINITE}]

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a further comment

According to Harvey Friedman's response to a question posted by the author on the Foundations of Mathematics (FOM) newsgroup, the statement \( \text{subclass}[\text{P}[x], \text{FINITE}] \) is also equivalent to the statement that \( x \) is finite when one assumes the axiom of regularity. In the \text{GOEDEL} program, the axiom of regularity is not automatically assumed to be true, but one could consider adding the hypothesis that \( x \) be regular. This matter will not be pursued further here.