intersections of subgroups of a group

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In[1]:= SetDirectory["l:"]; << goedel.11jun05a

:Package Title: goedel.11jun05a 2011 June 5 at 2:30 a.m.

Loading takes about eleven minutes, half that time due to builtin pauses.

It is now: 2011 Jun 6 at 14:52

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now: 2011 Jun 6 at 15:3

summary

The intersection of any nonempty collection of subgroups of a group is a subgroup. The key result needed to derive this fact is a rewrite rule that currently explicitly involves setpart.

In[2]:= Aclosure[image[IMAGE[oopart[x]], setpart[y]]]

Out[2]= image[IMAGE[oopart[x]], Aclosure[setpart[y]]]

a new conditional rewrite rule

To facilitate the derivation of Aclosure rules without having to always remember to use a setpart wrapper, a new conditional rewrite rule will be introduced.

Lemma. (Eliminating the setpart wrapper.)

In[6]:= Map[implies[member[y, z], #] &, SubstTest[implies, equal[y, setpart[t]], equal[Aclosure[image[IMAGE[oopart[x]], y]], image[IMAGE[oopart[x]], Aclosure[y]], t → y]] // Reverse

Out[6]= or[equal[Aclosure[image[IMAGE[oopart[x]], y]], image[IMAGE[oopart[x]], Aclosure[y]]], not[member[y, z]]] = True

In[7]:= or[equal[Aclosure[image[IMAGE[oopart[x_]], y_]], y_],
image[IMAGE[oopart[x_]], Aclosure[y_]], not[member[y_, z_]]] = True
Lemma. (Eliminating the oopart wrapper.)

```
In[11]:= SubstTest[implies, equal[x, oopart[t]],
    or[equal[Aclosure[image[IMAGE[x], y]], image[IMAGE[x], Aclosure[y]]],
      not[member[y, z]], t \[Function] x] // Reverse
```

```
Out[11]= or[equal[Aclosure[image[IMAGE[x], y]], image[IMAGE[x], Aclosure[y]]],
      not[FUNCTION[x]], not[FUNCTION[inverse[x]]], not[member[y, z]]] = True
```

```
In[12]:= or[equal[Aclosure[image[IMAGE[x_], y_]], image[IMAGE[x_], Aclosure[y_]]],
      not[FUNCTION[x_]], not[FUNCTION[inverse[x_]]], not[member[y_, z_]]] := True
```

Theorem. A conditional rewrite rule.

```
In[13]:= implies[and[ONEONE[x], member[y, V]],
    equal[Aclosure[image[IMAGE[x], y]], image[IMAGE[x], Aclosure[y]]]]
```

```
```

```
In[15]:= Aclosure[image[IMAGE[x_], y_]] :=
    image[IMAGE[x], Aclosure[y]] ; and[ONEONE[x], member[y, V]]
```

---

**derivation of an Aclosure rule for subgroups**

Theorem. Any collection of subsets of a group is determined by the collection of their domains.

```
In[16]:= SubstTest[image, IMAGE[composite[id[funpart[t]], inverse[FIRST]]],
    image[IMAGE[FIRST], intersection[y, P[funpart[t]]]], t \[Function] gp[x]] // Reverse
```

```
Out[16]= image[IMAGE[composite[id[gp[x]], inverse[FIRST]]],
    image[IMAGE[FIRST], intersection[y, P[gp[x]]]]] =
    intersection[y, P[gp[x]]]
```

```
In[17]:= image[IMAGE[composite[id[gp[x]], inverse[FIRST]]],
    image[IMAGE[FIRST], intersection[y_, P[gp[x]]]]] :=
    intersection[y, P[gp[x]]]
```

Theorem. The class of ranges of subgroups of a group is closed under arbitrary intersections.

```
In[18]:= SubstTest[Aclosure,
    intersection[binclosed[t], complement[P[complement[set[e[gp[x]]]]]]],
    fix[IMAGE[inv[gp[x]]]], t \[Function] gp[x]] // Reverse
```

```
Out[18]= Aclosure[image[IMAGE[SECOND], intersection[GROUPS, P[gp[x]]]]] =
    image[IMAGE[SECOND], intersection[GROUPS, P[gp[x]]]]
```

```
In[19]:= Aclosure[image[IMAGE[SECOND], intersection[GROUPS, P[gp[x]]]]] :=
    image[IMAGE[SECOND], intersection[GROUPS, P[gp[x]]]]
```

Theorem. The class of domains of subgroups of a group is closed under arbitrary intersections.
Theorem. The class of subgroups of a group is closed under arbitrary intersections.