similarity for ordinals

Johan G. F. Belinfante
2010 November 5

In[1]:= SetDirectory["l:"]; << goedel.10nov05a

:Package Title: goedel.10nov05a 2010 November 5 at 5:30 a.m.

It is now:  2010 Nov 5 at 12:50

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

summary

It is shown that the (two-sided) restriction of the subset relation \( S \) to one ordinal can not be similar to its restriction to a different ordinal.

abbreviation

The restriction of the subset relation to a class \( x \) will be abbreviated to \( s[x] \).

In[2]:= \( s[x_] := \text{composite}[\text{id}[x], S, \text{id}[x]] \)

domain

An elementary result about domains of bijections is derived here. Recall that \( \text{bij}[x, y] \) denotes the set of all bijections with domain \( x \) and range \( y \).

Lemma.

In[3]:= Map[not, SubstTest[and, implies[p1, p3], implies[and[p2, p3], p4],
not[implies[and[p1, p2], p4]], (p1 \rightarrow \text{member}[t, \text{bij}[x, y]], p2 \rightarrow \text{member}[x, z],
p3 \rightarrow \text{equal}[\text{domain}[t], x], p4 \rightarrow \text{member}[\text{domain}[t], z])] \// Reverse

Out[3]= or[\text{member}[\text{domain}[t], z], \text{not}[\text{member}[t, \text{bij}[x, y]]], \text{not}[\text{member}[x, z]]] \Rightarrow \text{True}

In[4]:= or[\text{member}[\text{domain}[t_], z_], \text{not}[\text{member}[t_, \text{bij}[x_, y_]]], \text{not}[\text{member}[x_, z_]]] := \text{True}

Theorem. If \( t \in \text{bij}[\text{ord}[x], y] \), then \( \text{domain}[t] \in \Omega \).
range

This section is analogous to the preceding one, but deals with ranges instead of domains.

Lemma.

Theorem. If \( t \in \text{bij}[x, \text{ord}[y]] \), then \( \text{range}[t] \in \Omega \).

Corollary.
Theorem. There can be no monotone bijection from one ordinal to a different ordinal.

Lemma. Monotonicity implies monotonicity of the inverse.

Lemma. The domain and range must be equal.

Theorem. There can be no monotone bijection from one ordinal to a different ordinal.
Eliminating the variable \( t \) yields the following restatement.

**Corollary.** If \( \text{bij}[\text{ord}[x], \text{ord}[y]] \cap \text{monotone}[S, S] \) is not empty, then \( \text{ord}[x] = \text{ord}[y] \).

**Corollary.** (Restatement without wrappers.)

In this section, the rigidity results are again restated, this time in terms of similarity of restrictions of the subset relation to ordinals.

**Theorem.** If \( s[\text{ord}[x]] \) is similar to \( s[\text{ord}[y]] \), then \( \text{ord}[x] = \text{ord}[y] \).

Since the converse also holds, the above result can be stated as a logical equivalence, which can be made into a simpler rewrite rule.
Corollary. A better rewrite rule.

\begin{verbatim}
In[27]:= equiv[member[pair[composite[id[ord[x]], S, id[ord[x]]],
composite[id[ord[y]], S, id[ord[y]]]], SIMILAR], equal[ord[x], ord[y]]]
Out[27]= True
\end{verbatim}

\begin{verbatim}
In[28]:= member[pair[composite[id[ord[x___]], S, id[ord[x___]]],
composite[id[ord[y___]], S, id[ord[y___]]]], SIMILAR] := equal[ord[x], ord[y]]
\end{verbatim}

Corollary. (Eliminating the \texttt{ord} wrappers.)

\begin{verbatim}
In[29]:= SubstTest[implies, and[equal[x, ord[u]], equal[y, ord[v]],
member[pair[s[x], s[y]], SIMILAR]], equal[x, y], {u \to x, v \to y}] // Reverse
Out[29]= or[equal[x, y], not[member[x, OMEGA]], not[member[y, OMEGA]], not[member[
pair[composite[id[x], S, id[x]], composite[id[y], S, id[y]]]], SIMILAR]] = True
In[30]:= or[equal[x___, y___], not[member[x___, OMEGA]], not[member[y___, OMEGA]], not[member[
pair[composite[id[x___], S, id[x___]], composite[id[y___], S, id[y___]]]], SIMILAR]] := True
\end{verbatim}

\section*{variable-free restatement}

A variable-free reformulation of the rigidity theorem derived in the preceding section can be obtained easily using \texttt{AssertTest}.

Theorem.

\begin{verbatim}
In[31]:= equal[composite[id[OMEGA], inverse[DUP], inverse[CART], inverse[IMAGE[id[S]]],
SIMILAR, IMAGE[id[S]], CART, DUP, id[OMEGA]], id[OMEGA]] // AssertTest
Out[31]= equal[composite[id[OMEGA], inverse[DUP], inverse[CART], inverse[IMAGE[id[S]]],
SIMILAR, IMAGE[id[S]], CART, DUP, id[OMEGA]], id[OMEGA]] = True
In[32]:= composite[id[OMEGA], inverse[DUP], inverse[CART], inverse[IMAGE[id[S]]],
SIMILAR, IMAGE[id[S]], CART, DUP, id[OMEGA]] := id[OMEGA]
\end{verbatim}

Corollary. The restriction of the similarity relation to the class of restrictions of the subset relation to ordinals is the identity relation on that class.

\begin{verbatim}
In[33]:= Map[composite[IMAGE[id[S]], CART, DUP, inverse[#]] &,
Assoc[composite[IMAGE[id[S]], CART, DUP],
composite[id[OMEGA], inverse[DUP], inverse[CART], inverse[IMAGE[id[S]]]],
composite[SIMILAR, IMAGE[id[S]], CART, DUP, id[OMEGA]]]] // Reverse
Out[33]= composite[id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]]],
SIMILAR, id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]]] =
id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]]
\end{verbatim}
In[34]:= composite[id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]]],
   SIMILAR, id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]] :=
   id[image[IMAGE[id[S]], image[CART, id[OMEGA]]]]