total orders and chains

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summary

A vertical section of the total order to[x] is a class of the form

\[ \text{class}[v, \text{member}[\text{pair}[u, v], t]] /. t \rightarrow \text{to}[x] \]

The class of all (small) vertical sections is

\[ \text{class}[v, \exists u, \text{and}[\text{member}[u, \text{fix}[t]], \text{equal}[v, \text{image}[t, \text{set}[u]]])] /. t \rightarrow \text{to}[x] \]

In this notebook it is shown that the class of all vertical sections of a total order is a chain with respect to inclusion. A succinct variable–free restatement is derived that connects the class TO of all (small) total orders with the class chains[S] of all (small) nests of sets.

derivation

Lemma. (Specialization of a general result that holds for all partial orders to the case of total orders.)

Corollary. (The same result for the inverse relation.)
Theorem. The vertical sections of a total order form a chain with respect to inclusion.

Corollary. Restatement without wrappers.

Lemma. A similar restatement, less transparent, but more suitable for the task of eliminating variables.

Theorem. (A succinct variable–free restatement.)

comment

An initial segment of a total order \( \leq \) is a class of the form \( \{x \mid x < y\} \). Vertical sections are of the form \( \{x \mid y \leq x\} \). Thus, an initial segment is the complement of a vertical section. For the usual total ordering of the class of all ordinals, the initial segments are sets, while the vertical sections are proper classes. In this section it is shown that for this reason, the chain considered in the above is empty for the usual ordering of the ordinals.
In[32]:= Map[not, SubstTest[implies, and[subclass[u, v], member[v, V]], member[u, V], 
    {u -> dif[OMEGA, ord[x]], v -> intersection[OMEGA, image[S, set[ord[x]]]]}]] // 
    Reverse

In[33]:= member[intersection[OMEGA, image[S, set[ord[x_]]]], V] := False

In[35]:= SubstTest[implies, equal[x, ord[t]], 
    not[member[intersection[OMEGA, image[S, set[x]]], V]], t -> x] // Reverse
Out[35]= or[not[member[x, OMEGA]], not[member[intersection[OMEGA, image[S, set[x]]], V]]] = True

In[36]:= or[not[member[x_, OMEGA]], 
    not[member[intersection[OMEGA, image[S, set[x_]]], V]]] := True

Lemma.

In[44]:= equal[intersection[OMEGA, complement[P[OMEGA]]], 0]
Out[44]= True

In[46]:= intersection[OMEGA, complement[P[OMEGA]]] := 0

Lemma.

In[47]:= domain[composite[VERTSECT[composite[id[OMEGA], S]], id[OMEGA]]] // Normality // Reverse
Out[47]= intersection[OMEGA, fix[composite[inverse[S], 
    BIGCAP, id[P[OMEGA]], S, VERTSECT[composite[id[OMEGA], S]]]]] = 
intersection[OMEGA, domain[VERTSECT[composite[id[OMEGA], S]]]]

In[48]:= % /. Equal -> SetDelayed

Lemma.

In[49]:= Map[equal[V, #] &, SubstTest[class, x, 
    or[not[member[x, w]], not[member[intersection[w, image[S, set[x]]], V]]], w -> OMEGA]]
Out[49]= equal[0, intersection[OMEGA, domain[VERTSECT[composite[id[OMEGA], S]]]]] = True

In[51]:= intersection[OMEGA, domain[VERTSECT[composite[id[OMEGA], S]]]] := 0

Theorem.

In[52]:= SubstTest[composite, x, id[domain[x]], 
    x -> composite[VERTSECT[composite[id[OMEGA], S]], id[OMEGA]]]
Out[52]= composite[VERTSECT[composite[id[OMEGA], S]], id[OMEGA]] = 0

In[53]:= composite[VERTSECT[composite[id[OMEGA], S]], id[OMEGA]] := 0

Corollary.

In[55]:= ImageComp[VERTSECT[composite[id[OMEGA], S]], id[OMEGA], V] // Reverse
Out[55]= image[VERTSECT[composite[id[OMEGA], S]], OMEGA] = 0

In[56]:= image[VERTSECT[composite[id[OMEGA], S]], OMEGA] := 0
The chain \texttt{image[VERTSECT[t], fix[t]]} is therefore empty for the relation \texttt{restrict[S,OMEGA,OMEGA]}.

\texttt{In[57]:= image[VERTSECT[t], fix[t]] /. t \rightarrow restrict[S, OMEGA, OMEGA]}

\texttt{Out[57]= 0}