characterizing ordinals using chain unions

Johan G. F. Belinfante
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In[1]:= SetDirectory["l:"]; << goedel.11jul17a

:Package Title: goedel.11jul17a 2011 July 17 at 4:50 a.m.
Loading takes about eleven minutes, half that time due to buildin pauses.
It is now: 2011 Jul 19 at 17:56
Loading Simplification Rules
TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3
weightlimit = 40
Loading completed.
It is now: 2011 Jul 19 at 18:6

summary

Ernst Zermelo in 1915 characterized ordinals as sets that satisfy \( \text{image}[\text{Succ}, x] \subseteq \text{succ}[x] \) and \( \text{Uclosure}[x] \subseteq \text{succ}[x] \).

In[2]:= \text{implies}[\text{member}[x, V], \text{equiv}[\text{member}[x, \text{OMEGA}], \text{and}[
\text{subclass}[\text{image}[\text{Succ}, x], \text{succ}[x]], \text{subclass}[\text{Uclosure}[x], \text{succ}[x]]]]] \quad \text{not} \quad \text{not} \n

This characterization was derived 2010 June 11 using the GOEDEL program in the posted notebook \texttt{zer-on-3.nb}. An analogous result is derived in this notebook, but with \text{Uchains} in place of \text{Uclosure}. Both results are independent of the axiom of regularity.

derivation

The following elementary theorem will be needed several times.

Theorem. If \( x \in y \) and \( y \subseteq \text{succ}[z] \), then \( x \in \text{succ}[z] \).

In[3]:= \text{SubstTest}[\text{implies}, \text{and}[\text{member}[x, y], \text{subclass}[y, t]], \text{member}[x, t], t \rightarrow \text{succ}[z]] \quad \text{Reverse} \n
Out[3]= or[\text{equal}[x, z], \text{member}[x, z], \text{not}[\text{member}[x, y]], \text{not}[\text{subclass}[y, \text{succ}[z]]]] = True
The easy half of the equivalence is that every ordinal satisfies \( \text{image}[\text{SUCC}, x] \subseteq \text{succ}[x] \) and \( \text{Uchains}[x] \subseteq \text{succ}[x] \). The former inclusion is already available in the \text{GOEDEL} program, while the latter is available using the \text{ord[x]} wrapper. The following theorem provides an unwrapped version of this fact.

Theorem. If \( x \in \Omega \), then \( \text{Uchains}[x] \subseteq \text{succ}[x] \).

For completeness, a slight variant of the above result is given in the following theorem.

Theorem. If \( x \in \Omega \), then \( \text{Uchains}[x] = \text{succ}[\text{U}[x]] \).

Comment. The class \( \Omega \) of all ordinals also satisfies \( \text{image}[\text{SUCC}, x] \subseteq \text{succ}[x] \) and \( \text{Uchains}[x] \subseteq \text{succ}[x] \). In this case, one can simplify the inclusions since \( \text{succ}[x] = x \) when \( x \) is a proper class. The following converse is due to Sion and Willmott.

Reference.

This result of Sion and Willmott was derived 2008 April 1 using the \text{GOEDEL} program in the posted notebook \text{uch-on.nb}.

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**a useful rewrite rule**

In this section a rewrite rule needed later is derived.

Lemma.

In this section a rewrite rule needed later is derived.
The class \( \bigcup (\cap (\Omega \cap x)) \) is an ordinal if and only if \( \Omega \) and \( x \) are not disjoint.

\[
\text{equiv}[\text{member}[\text{U}[\text{intersection}[\Omega, x]], \Omega], \\
\text{not}[\text{equal}[0, \text{intersection}[\Omega, x]]]]
\]


\[
\text{member}[\text{U}[\text{intersection}[\Omega, x]], \Omega] := \text{not}[\text{equal}[0, \text{intersection}[\Omega, x]]]
\]

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**the main theorem**

Since any set of ordinals is a chain, the union of any subset \( x \subseteq \Omega \cap y \) belongs to \( \text{Uchains}[y] \).

Theorem. If \( x \in \text{P}[\Omega \cap y] \), then \( \text{U}[x] \in \text{Uchains}[y] \).

\[
\text{Map}[\text{not}, \text{SubstTest}[\text{and}, \text{implies}[\text{and}[p0, p1, p3], p4], \\
\text{implies}[p2, p3], \text{not}[\text{implies}[\text{and}[p0, p1, p2], p4]], \\
\{p0 \rightarrow \text{member}[x, z], p1 \rightarrow \text{subclass}[x, y], p2 \rightarrow \text{subclass}[x, \Omega], \\
p3 \rightarrow \text{subclass}[\text{P}[x], \text{chains}[S]], p4 \rightarrow \text{member}[\text{U}[x], \text{Uchains}[y]]\}] // \text{Reverse}
\]

Out[15]= or[\text{member}[\text{U}[x], \text{Uchains}[y]], \text{not}[\text{member}[x, z]], \\
\text{not}[\text{subclass}[x, \Omega]], \text{not}[\text{subclass}[x, y]]] = \text{True}

\[
\text{or}[\text{member}[\text{U}[x], \text{Uchains}[y]], \text{not}[\text{member}[x, z]], \\
\text{not}[\text{subclass}[x, \Omega]], \text{not}[\text{subclass}[x, y]]] := \text{True}
\]

The abbreviation \( t = \cap (\Omega - x) \) will be used in the following for the least ordinal that does not belong to \( x \). Applying the above theorem to this subset of \( x \) yields the following.

Corollary.

\[
\text{SubstTest}[\text{or}, \text{member}[\text{U}[t], \text{Uchains}[x]], \text{not}[\text{member}[t, V]], \\
\text{not}[\text{subclass}[t, \Omega]], \text{not}[\text{subclass}[t, x]], t \rightarrow \text{A}[\text{dif}[\Omega, x]]] // \text{Reverse}
\]

Out[17]= or[\text{member}[\text{U}[\text{intersection}[\Omega, \text{complement}[x]]], \text{Uchains}[x]], \text{subclass}[\Omega, x]] = \text{True}

\[
(\% /. x \rightarrow x_\_] /. \text{Equal} \rightarrow \text{SetDelayed}
\]

Lemma. If \( \text{Uchains}[x] \subseteq \text{succ}[x] \) and \( t = \cap (\Omega - x) \) is a set, then \( \text{U}[t] \in \text{succ}[x] \).
Lemma. If $x = \text{U}[t]$, then $x \in \Omega$ or $\Omega \subseteq x$.

Lemma. If $\text{Uchains}[x] \subseteq \text{succ}[x]$ and $\text{U}[t] \in x$, then $x \in \Omega$ or $\Omega \subseteq x$.

Lemma. If $x = \text{succ}[\text{U}[t]]$ then either $x \in \Omega$ or $\Omega \subseteq x$.

Lemma. A consequence of the hypothesis $\text{image}[	ext{SUCC}, x] \subseteq \text{succ}[x]$.
$\text{Main Theorem}$. If $\text{image(SUCC, x)} \subseteq \text{succ(x)}$ and $\text{Uchains(x)} \subseteq \text{succ(x)}$, then either $x \in \Omega$ or $\Omega \subseteq x$.

$\text{Lemma}$. If $t = \bigcap (\Omega - x)$ is an ordinal, either $U[t] = t$ or $\text{succ(U[t])} = t$.

$\text{Corollary}$. Any set $x$ satisfying $\text{image(SUCC, x)} \subseteq \text{succ(x)}$ and $\text{Uchains(x)} \subseteq \text{succ(x)}$ is an ordinal.
a variable-free formulation

In this section a variable-free formula is derived that combines the main theorem with its converse.

Theorem. A membership rule for the class of all sets satisfying \( \text{Uchains}[x] \subset \text{succ}[x] \).

Lemma. A formula for the class of ordinals.
In[41]:= SubstTest[and, subclass[u, v], subclass[v, u],
    {u -> intersection[fix[composite[inverse[SUCC]], S, UCHAINS]],
     fix[composite[inverse[SUCC]], S, IMAGE[SUCC]]}, v -> OMEGA]

Out[41]= equal[OMEGA, intersection[fix[composite[inverse[SUCC]], S, UCHAINS]],
     fix[composite[inverse[SUCC]], S, IMAGE[SUCC]]] = True

In[42]:= intersection[fix[composite[inverse[SUCC]], S, UCHAINS]],
     fix[composite[inverse[SUCC]], S, IMAGE[SUCC]]] := OMEGA