a characterization of OMEGA

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In[1]:= SetDirectory["l:"]; << goedel97.06a; << tools.m

:Package Title: goedel97.06a                      2007 September 6 at 3:20 p.m.
It is now:  2007 Sep 6 at 21:14
Loading Simplification Rules

TOOLS.M               Revised 2007 June 25

weightlimit = 40

summary

The class OMEGA: of all ordinal numbers is successor-invariant and closed under arbitrary unions.

In[2]:= Uclosure[OMEGA]

Out[2]= OMEGA

In[3]:= invariant[SUC, OMEGA]


These two properties can be used to characterize the class of ordinals. It is shown in this notebook that the class OMEGA is the smallest class which is successor-invariant and closed under arbitrary unions. Comment: This result holds independently of the axiom of regularity.

derivation

Lemma. (For convenience, the temporary abbreviation t is used here for the least ordinal that does not belong to x. This temporary variable is eliminated from the final statement of the lemma.)
Main Theorem. The class of ordinals is the least class that is successor-invariant and closed under arbitrary unions.

Lemma.\[\text{not} \left( \text{equal} \left[ \text{core} \left[ x, U \left[ A \left[ \text{intersection} \left[ \Omega, \text{complement} \left[ x \right] \right] \right] \right] \right] \right], x \right) \], 
\text{not} \left( \text{equal} \left[ x, U \left[ \text{closure} \left[ x \right] \right] \right] \right), \text{subclass} \left[ \Omega, x \right] = \text{True} \]

Lemma.\[\text{not} \left( \text{equal} \left[ x, U \left[ \text{closure} \left[ x \right] \right] \right] \right), \text{subclass} \left[ \Omega, x \right] = \text{True} \]

Lemma.\[\text{not} \left( \text{equal} \left[ x, U \left[ \text{closure} \left[ x \right] \right] \right] \right), \text{subclass} \left[ \Omega, x \right] = \text{True} \]

Main Theorem. The class of ordinals is the least class that is successor-invariant and closed under arbitrary unions.
Corollary. There is no set that is successor-invariant and closed under arbitrary unions.

\[
\text{Map}[\text{equal}[V, \#] \& \text{SubstTest}[\text{class}, x, \text{implies}[\text{member}[x, u], \text{subclass}[v, x]],
\{u \rightarrow \text{intersection}[\text{invar}[\text{SUCC}], \text{fix}[\text{UCLOSURE}], v \rightarrow \text{OMEGA}]\}]
\]

Out[12]= equal[0, \text{intersection}[\text{fix}[\text{UCLOSURE}], \text{invar}[\text{SUCC}]]] = True

\[
\text{intersection}[\text{fix}[\text{UCLOSURE}], \text{invar}[\text{SUCC}]] := 0
\]

comments and acknowledgments

The theorem in this notebook was proposed (on page 208) as a definition of the class of ordinal numbers by Jan Mycielski.


A result somewhat similar to the final corollary in the preceding section was derived on 2005 January 15 in another notebook. Benjamin Lamothe, then a student at the Massachusetts Institute of Technology, and the present author used the \text{GOEDEL} program to derive what is commonly called Hilbert's Paradox (1905): there is no set that is closed under arbitrary unions and the power set operation:

\[
\text{disjoint}[\text{fix}[\text{UCLOSURE}], \text{invar}[\text{POWER}]]
\]

Out[14]= True