unital subsemigroups

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In[1] := SetDirectory[" l:"]; << go del.11may21a

:Package Title: go del.11may21a
2011 May 21 at 12:20 noon

Loading takes about ten minutes, half that time due to builtin pauses.

It is now: 2011 May 22 at 19:20

Loading Simplification Rules

TOOLS.M is now incorporated in the GOEDEL program as of 2010 September 3

weightlimit = 40

Loading completed.

It is now: 2011 May 22 at 19:31

summary

A subsemigroup \( x \) of a binary operation \( y \) is said to be unital if \( y \) has a neutral (or unity) element \( e[y] \) and this element belongs to \( \text{fix}[	ext{domain}[x]] \). In this notebook various theorems are derived about unital subsemigroups of monoids and groups.

ids\([x]\)

From the definition of the class \( \text{ids}[x] \) of neutral (or identity) elements, one obtains the following characterization.

Lemma. Characterization of neutral elements.

In[2] := SubstTest[member, \( x \), intersectio n\( u, v, w \),
   \{u \rightarrow \text{fix}[\text{domain}[y]],
    v \rightarrow \text{complement}[\text{domain}[\text{fix}[\text{composite}[\text{inverse}[\text{SECOND}], \text{Di}, y]]]],
    w \rightarrow \text{complement}[\text{range}[\text{fix}[\text{composite}[\text{inverse}[\text{FIRST}], \text{Di}, y]]]]\}]

Out[2]= and[member[x, \text{fix}[\text{domain}[y]]], subclass[composite[y, \text{LEFT}[x]], \text{Id}],
    subclass[composite[y, \text{RIGHT}[x]], \text{Id}]] = member[x, \text{ids}[y]]

In[3] := and[member[x_, \text{fix}[\text{domain}[y_]]], subclass[composite[y_, \text{LEFT}[x_]], \text{Id}],
    subclass[composite[y_, \text{RIGHT}[x_]], \text{Id}]] := member[x, \text{ids}[y]]

Corollary.
A binary operation has at most one neutral element.

If \( \text{binop}[x] \) has a neutral element, then it is \( \text{e}[\text{binop}[x]] \in \text{fix}[	ext{domain}[	ext{binop}[x]]] \), and if not, then \( \text{e}[\text{binop}[x]] = \text{V} \).

Theorem. A simplification rule.

A monoid \( x \) is a semigroup with a neutral element \( e[x] \).

Lemma.
The neutral element of a monoid is idempotent.

A semigroup can only have a unital subsemigroup if it is itself a monoid.

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Corollary. Any unital semigroup of a semigroup is a monoid.

Corollary. Any unital semigroup of a monoid is a submonoid.

unital subsemigroups of a group

A group is a monoid that is also a quasigroup. Any theorem about monoids is therefore true in particular for groups.

Theorem. Any unital subsemigroup of a group is a monoid.
Theorem. For a unital subsemigroup of a group, the identity element of the group is the identity element of the semigroup.

unital binary closed sets

A class y that is binary closed under a binary operation x is said to be unital if e[x] ∈ y. In this section it is shown that the restriction of a group to the cartesian square of a unital binary closed class is a submonoid.

Lemma.

Theorem. If x is a group, if y is binary closed under x and if e[x] ∈ y, then the restriction of x to y × y is a monoid.
In this final section some results derived earlier about neutral elements are specialized to the case of groups.

Lemma. (Specialization to groups by introducing gp wrappers.)

In[52]:= \text{SubstTest}[\text{or}, \text{member}[x, \text{ids}[t]], \\
not[\text{member}[x, \text{fix}[\text{domain}[t]]]], \\
not[\text{subclass}[\text{composite}[t, \text{LEFT}[x]], \text{Id}]], \\
not[\text{subclass}[\text{composite}[t, \text{RIGHT}[x]], \text{Id}]], t \rightarrow \text{gp}[y]] \text{// Reverse}

Out[52]= \text{or}[\text{equal}[x, e[\text{gp}[y]]], \text{not}[\text{member}[x, \text{range}[\text{gp}[y]]]]], \\
\text{not}[\text{subclass}[\text{composite}[\text{gp}[y], \text{LEFT}[x]], \text{Id}]], \\
\text{not}[\text{subclass}[\text{composite}[\text{gp}[y], \text{RIGHT}[x]], \text{Id}]]] = \text{True}

In[53]:= \text{or}[\text{equal}[x_, e[\text{gp}[y_]]], \text{not}[\text{member}[x_, \text{range}[\text{gp}[y_]]]]], \\
\text{not}[\text{subclass}[\text{composite}[\text{gp}[y_], \text{LEFT}[x_]], \text{Id}]], \\
\text{not}[\text{subclass}[\text{composite}[\text{gp}[y_], \text{RIGHT}[x_]], \text{Id}]]] := \text{True}

Theorem.

In[54]:= \text{SubstTest}[\text{implies}, \text{equal}[x, \text{gp}[t]], \\
\text{subclass}[\text{composite}[x, \text{LEFT}[e[x]], \text{Id}], t \rightarrow x] \text{// Reverse} / \text{// MapNotNot}

Out[54]= \text{or}[\text{not}[\text{member}[x, \text{GROUPS}]], \text{subclass}[\text{composite}[x, \text{LEFT}[e[x]]], \text{Id}]] = \text{True}

In[55]:= \text{or}[\text{not}[\text{member}[x_, \text{GROUPS}]], \text{subclass}[\text{composite}[x_, \text{LEFT}[e[x_]]], \text{Id}]] := \text{True}

Dual Theorem.

In[56]:= \text{SubstTest}[\text{implies}, \text{equal}[x, \text{gp}[t]], \\
\text{subclass}[\text{composite}[x, \text{RIGHT}[e[x]], \text{Id}], t \rightarrow x] \text{// Reverse} / \text{// MapNotNot}

Out[56]= \text{or}[\text{not}[\text{member}[x, \text{GROUPS}]], \text{subclass}[\text{composite}[x, \text{RIGHT}[e[x]]], \text{Id}]] = \text{True}

In[57]:= \text{or}[\text{not}[\text{member}[x_, \text{GROUPS}]], \text{subclass}[\text{composite}[x_, \text{RIGHT}[e[x_]]], \text{Id}]] := \text{True}