Zermelo ordinals are regular

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summary

John von Neumann is generally credited with the idea of defining ordinals recursively by the condition that each ordinal is the class of all preceding ones.

In 1941 Paul Bernays remarked that Ernst Zermelo in 1915 (in an unpublished notebook) had anticipated this idea, defining ordinals as sets that satisfy the two properties \( \text{Uclosure}_x \subseteq \text{succ}_x \) and \( \text{image}_x[S\text{ucc}, x] \subseteq \text{succ}_x \). Bernays points out, however, that Zermelo in 1915 had lacked sufficient axioms to prove the existence of certain needed classes to carry out this program.

Bernays says that Zermelo had added a third condition \( 0 \notin x \) to rule out zero as an ordinal. Bernays suggested weakening this third condition to \( 0 = x \) or \( 0 \in x \), to permit inclusion of 0 as an ordinal. Willard van Orman Quine in a footnote on page 157 in his book points out that the condition \( 0 = x \) or \( 0 \in x \) can be omitted altogether as it follows from the condition \( \text{Uclosure}_x \subseteq \text{succ}_x \).

One interesting feature of Zermelo’s definition is that one does not require the axiom of regularity. It will be shown in this notebook that any set satisfying Zermelo’s two conditions belongs to the remarkable class \text{REGULAR}, defined as follows:
In[5]: = class[x, forall y, implies[member[x, y], exists[z, and[member[z, y], disjoint[y, z]]]]]
Out[5] = REGULAR

The GOEDEL program does not assume that the axiom of regularity holds. The axiom of regularity is equivalent to the statement that every set belongs to REGULAR.

In[6]: = equal[REGULAR, V]
Out[6] = AxReg

John Isbell defined ordinals as sets satisfying \( P[x] \subset \text{succ}[x] \). Isbell’s condition also does not require the axiom of regularity.

In[7]: = class[x, subclass[intersection(FULL, P[x]), succ[x]]]
Out[7] = OMEGA

In[8]: = "John Rolfe Isbell, A definition of ordinal numbers

In the GOEDEL program, as well as in the author’s Otter work, ordinals were defined using John Isbell’s definition.


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**almost successor-invariant**

For convenience, a class \( x \) satisfying the condition \( \text{image}\{\text{SUCC}, x\} \subset \text{succ}[x] \) will be called **almost successor invariant**. Any ordinal in Isbell’s sense has this property:

In[10]: = implies[member[x, OMEGA], subclass[member[x, SUCC], succ[x]]]
Out[10] = True

The class of almost-successor-invariant sets is:

In[11]: = (class[x, subclass[member[t, x], succ[x]]] /. t -> SUCC) // InvertFix

Theorem. Membership rule for the class of almost successor invariant sets.

In[12]: = member[x, fix[composite[inverse[SUCC], S, IMAGE[SUCC]]]] // AssertTest
Out[12] = member[x, fix[composite[inverse[SUCC], S, IMAGE[SUCC]]]] =
    and[member[x, V], subclass[member[succ[x], Succ], Succ[x]]]

In[13]: = member[x_, fix[composite[inverse[SUCC], S, IMAGE[SUCC]]]] :=
    and[member[x, V], subclass[member[Succ, Succ], Succ[x]]]
Observation. The class \textbf{OMEGA} of Isbell ordinals is contained in the class of almost successor invariant sets.

\begin{verbatim}
In[14]:= subclass[OMEGA, fix[composite[inverse[SUCC], S, IMAGE[SUCC]]]]
Out[14]= True
\end{verbatim}

\section*{almost Uclosed}

For any class \(x\), the class of all unions of subsets of \(x\) is called \textbf{Uclosure}[x].

\begin{verbatim}
In[15]:= class[u, exists[s, andsubclass[s, x], equal[u, U[s]]]]
Out[15]= Uclosure[x]
\end{verbatim}

For convenience, a class \(x\) satisfying the condition \(\text{Uclosure}[x] \subset \text{succ}[x]\) will be called \textbf{almost Uclosed}. Any ordinal in Isbell's sense has this property:

\begin{verbatim}
In[16]:= implies[member[x, OMEGA], subclass[Uclosure[x], succ[x]]]
Out[16]= True
\end{verbatim}

Lemma. Quine's comment that almost Uclosed sets automatically satisfy the Bernays condition \(0 = x\ or\ 0 \in x\).

\begin{verbatim}
In[17]:= SubstTest[implies, and[member[t, u], subclass[u, v]],
member[t, v], \{t \rightarrow 0, u \rightarrow \text{Uclosure}[x], v \rightarrow \text{succ}[x]\}] // Reverse
Out[17]= or[equal[0, x], member[0, x], not[subclass[Uclosure[x], succ[x]]]] = True
In[18]:= or[equal[0, x_], member[0, x_], not[subclass[Uclosure[x_], succ[x_]]]] := True
\end{verbatim}

The class of almost-Uclosed sets is:

\begin{verbatim}
In[19]:= class[x, subclass[Uclosure[x], succ[x]]]
Out[19]= fix[composite[inverse[SUCC], S, UCLOSURE]]
\end{verbatim}

Theorem. Membership rule for the class of almost-Uclosed sets.

\begin{verbatim}
In[20]:= member[x, fix[composite[inverse[SUCC], S, UCLOSURE]]] // AssertTest
Out[20]= member[x, fix[composite[inverse[SUCC], S, UCLOSURE]]]
and[member[x, V], subclass[Uclosure[x], succ[x]]]
In[21]:= member[x_, fix[composite[inverse[SUCC], S, UCLOSURE]]] :=
and[member[x, V], subclass[Uclosure[x], succ[x]]]
\end{verbatim}

Theorem. The class \textbf{OMEGA} of Isbell ordinals is contained in the class of almost-Uclosed sets.
rules about REGULAR

Theorem.

In[24]:= Map[not, SubstTest[implies, subclass[t, REGULAR],
not[member[t, t]], t → U[intersection[REGULAR, x]]]] // Reverse
Out[24]= member[U[intersection[REGULAR, x]], U[intersection[REGULAR, x]]] = False

In[25]:= member[U[intersection[REGULAR, x_]], U[intersection[REGULAR, x_]]] := False

Lemma.

In[26]:= SubstTest[implies, member[succ[u], v], member[u, U[v]],
{u → U[intersection[REGULAR, x]], v → intersection[REGULAR, x]]} // Reverse
Out[26]= or[not[member[intersection[REGULAR, x], V]],
not[member[succ[U[intersection[REGULAR, x]], x]], x]] = True

In[27]:= (% /_. x → x_ )/. Equal -> SetDelayed

Lemma.

In[28]:= SubstTest[implies, member[succ[t], x],
member[t, V], t → U[intersection[REGULAR, x]]] // Reverse
Out[28]= or[member[intersection[REGULAR, x]], V],
not[member[succ[U[intersection[REGULAR, x]], x]], x]] = True

In[29]:= (% /_. x → x_ )/. Equal -> SetDelayed

Theorem.

In[30]:= Map[not, SubstTest[and, implies[p1, p2], implies[p1, not[p2]],
{p1 → member[succ[U[intersection[REGULAR, x]], x]], p2 → member[intersection[REGULAR, x], V]]}]

In[31]:= member[succ[U[intersection[REGULAR, x_]], x_]] := False
a core rule

Lemma.

In[32]:=  SubstTest[implies, equal[x, union[t, y]], equal[core[x, U[y]], U[y]], t \to x] // Reverse
Out[32]=  or[equal[core[x, U[y]], U[y]], notsubclass[y, x]] := True

In[33]:=  or[equal[core[x, U[y]], U[y]], notsubclass[y, x]] := True

Theorem.

In[34]:=  SubstTest[or, equal[core[x, U[t]], U[t]], notsubclass[t, x], t \to intersection[x, y]] // Reverse
Out[34]=  equal[core[x, U[intersection[x, y]]], U[intersection[x, y]]] := True

In[35]:=  core[x, U[intersection[x, y]]] := U[intersection[x, y]]

Zermelo \Rightarrow regular

A set that is almost Uclosed and almost successor invariant will be called a Zermelo number. In this section it is shown that Zermelo numbers belong to the class \textbf{REGULAR}.

Lemma.

In[37]:=  SubstTest[implies, and[member[u, v], subclass[v, w]], member[u, w], \{u \to U[intersection[setpart[x], REGULAR]], v \to Uclosure[setpart[x]], w \to succ[setpart[x]]\}] // Reverse
Out[37]=  or[subseteq[setpart[x], U[intersection[REGULAR, setpart[x]]]], member[intersection[REGULAR, setpart[x]], setpart[x]], notsubclass[Uclosure[setpart[x]], succ[setpart[x]]]] := True

In[38]:=  (\% /. x \to x_\_ ) /. Equal \to SetDelayed

Corollary. If an almost-Uclosed set \(x\) is not regular, then \(U[\text{REGULAR} \cap x] \notin x\).

In[39]:=  Map[not, SubstTest[and, implies[p1, or[p3, p4]], implies[p2, not[p3]], not[implies[and[p1, p2], p4]], \{p1 \to subclass[Uclosure[setpart[x]], succ[setpart[x]]], p2 \to not[member[setpart[x], REGULAR]], p3 \to equal[setpart[x], U[intersection[REGULAR, setpart[x]]]], p4 \to member[U[intersection[REGULAR, setpart[x]]], setpart[x]]\}] // Reverse
Out[39]=  or[member[setpart[x], REGULAR], member[intersection[REGULAR, setpart[x]], setpart[x]], notsubclass[Uclosure[setpart[x]], succ[setpart[x]]]] := True
Lemma.

SubstTest[implies, and[member[u, v], subclass[v, w]],
  member[u, w], [u \to U[intersection[REGULAR, setpart[x]]]],
  v \to setpart[x], w \to image[inverse[SUCC], succ[setpart[x]]]] // Reverse

Corollary.  (Restatement, removing the setpart wrapper.)

A variable-free formulation of this result follows.

Theorem.  The class of Zermelo numbers is contained in the class of regular sets.

Corollary.  (Restatement, removing the setpart wrapper.)

A variable-free formulation of this result follows.

Theorem.  The class of Zermelo numbers is contained in the class of regular sets.
In[48]:= subclass[intersection[fix[composite[inverse[SUCC], S, UCLOSURE]],
fix[composite[inverse[SUCC], S, IMAGE[SUCC]]], REGULAR] := True