the equations \( \text{Id} = \text{IMAGE}[x] \) and \( V = \text{fix}[\text{IMAGE}[x]] \)

Johan G. F. Belinfante
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**summary**

The equations \( \text{Id} = \text{IMAGE}[x] \) and \( V = \text{fix}[\text{IMAGE}[x]] \) are solved in this notebook.

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**the equation \( \text{Id} = \text{IMAGE}[x] \)**

Lemma.

\[
\text{In}[2]:= \text{SubstTest}[\text{implies}, \text{equal}[u, v], \text{equal}[\text{domain}[u], \text{domain}[v]], \{u \rightarrow \text{Id}, v \rightarrow \text{IMAGE}[x]\}]
\]

\[
\text{Out}[2] := \text{or}[\text{equal}[V, \text{domain}[\text{VERTSECT}[x]]], \text{not}[\text{equal}[\text{Id}, \text{IMAGE}[x]]]] = \text{True}
\]

\[
\text{In}[3]:= (\% /. x \rightarrow x_\_ \_). \text{Equal} \rightarrow \text{SetDelayed}
\]

Lemma.

\[
\text{In}[4]:= \text{SubstTest}[\text{implies}, \text{equal}[u, v], \text{equal}[\text{image}[w, u], \text{image}[w, v]], \{u \rightarrow \text{Id}, v \rightarrow \text{IMAGE}[x], w \rightarrow \text{cross}\[\text{inverse}[\text{SINGLET}]\], \text{inverse}[E]]\}]
\]

\[
\text{Out}[4] := \text{or}[\text{equal}[\text{Id}, \text{thindpart}[x]], \text{not}[\text{equal}[\text{Id}, \text{IMAGE}[x]]]] = \text{True}
\]

\[
\text{In}[5]:= (\% /. x \rightarrow x_\_ \_ \_). \text{Equal} \rightarrow \text{SetDelayed}
\]

Lemma.
These three lemmas yield a necessary condition for \( x \) to be a solution of the equation \( \text{Id} = \text{IMAGE}[x] \).

This condition is also sufficient:

Combining these two implications yields a simple rewrite rule for the solution of the equation \( \text{Id} = \text{IMAGE}[x] \).
The reverse implication also holds.

Combining these two implications yields a rewrite rule for the solution of the equation \( V = \text{fix}[\text{IMAGE}[x]] \).

some falsehoods

Generally speaking it does not seem desirable to add rules to the effect that various statements are false because there would be too many such rules, and many of them would not be very interesting. A few such rules are useful to provide counterexamples, and will be added on a case by case basis as interesting applications for them are discovered. In this notebook two basic falsehoods are needed. Both of these should be made permanent. The first one says that the membership relation is not equal to the identity function:

The second one says that the subset relation is not equal to the identity function:
To avoid having to add similar rewrite rules for the inverses of \( E \) and \( S \), the a single rule for inverses will be derived. A temporary rule is needed:

\[
\begin{align*}
\text{In [24]} & : \quad \text{equal} [\text{composite} [\text{Id}, x], \text{Id}] \text{ // AssertTest // Reverse} \\
\text{Out [24]} & : \quad \text{and} \left[ \text{equal} [V, \text{fix}[x]], \text{subclass} [\text{composite} [\text{Id}, x], \text{Id}] \right] = \\
& \quad \text{equal} [\text{Id}, \text{composite} [\text{Id}, x]] \\
\text{In [25]} & : \quad \text{and} \left[ \text{equal} [V, \text{fix}[x_-]], \text{subclass} [\text{composite} [\text{Id}, x_-], \text{Id}] \right] := \\
& \quad \text{equal} [\text{Id}, \text{composite} [\text{Id}, x]]
\end{align*}
\]

The following will be made permanent:

\[
\begin{align*}
\text{In [26]} & : \quad \text{equal} [\text{Id}, \text{inverse} [x]] \text{ // AssertTest} \\
\text{Out [26]} & : \quad \text{equal} [\text{Id}, \text{inverse} [x]] = \text{equal} [\text{Id}, \text{composite} [\text{Id}, x]] \\
\text{In [27]} & : \quad \text{equal} [\text{Id}, \text{inverse} [x_-]] := \text{equal} [\text{Id}, \text{composite} [\text{Id}, x]]
\end{align*}
\]

two examples

In this section, two examples are considered. The first case is now automatic, and does not need a special rule.

\[
\begin{align*}
\text{In [28]} & : \quad \text{equal} [V, \text{fix}[\text{IMAGE} [\text{inverse} [S]]]] \\
\text{Out [28]} & : \quad \text{False}
\end{align*}
\]

Here is another example. This case does require a special rule:

\[
\begin{align*}
\text{In [29]} & : \quad \text{SubstTest} [\text{equal}, V, \text{fix}[\text{IMAGE} [x]], x \rightarrow \text{inverse} [E]] \\
\text{Out [29]} & : \quad \text{equal} [V, \text{fix}[\text{BIGCUP}]] = \text{False} \\
\text{In [30]} & : \quad \text{equal} [V, \text{fix}[\text{BIGCUP}]] := \text{False}
\end{align*}
\]